EQUALITY OF EDGE DOMINATION AND CONNECTED EDGE DOMINATION IN GRAPHS

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Abstract. Let G be a (p,q) –graph with edge domination number γ' and connected edge domination number γ'_c. In this paper we investigate the structure of graphs in which some of the edge domination parameters are equal. We characterize connected graphs for which $\gamma' = \gamma'_c$.

1. Introduction

By a graph G = (V,E) we mean a finite undirected graph without loops or multiple edges. Terms not defined here are used in the sense of Harary [1].

The concept of edge domination was introduced by Mitchell and Hedetniemi. A subset X of E is called an edge dominating set of G if every edge not in X is adjacent to some edge in X. The edge domination number $\gamma'(G)$ (or γ' for short) of G is the minimum cardinality taken over all edge dominating sets of G. An edge dominating set X of is called a connected edge dominating set of G if the induced subgraph $\langle X \rangle$ is connected. The connected edge domination number $\gamma'_c(G)$ (or γ'_c for short) of G is the minimum cardinality taken over all connected edge dominating sets of G.

Allan and Laskar [2] proved that for any $K_{1,3}-$ free graph, the domination number and independent domination number are equal. Topp and Volkmann [3] generalized the result of Allan and Laskar and constructed new classes of graphs with equal domination and independent domination number.

Harary and Livingston [4] characterized caterpillars with equal domination and independent domination number. In [5] they gave the characterization of trees with equal domination and independent domination number.

Payan and Xuong [6] proved that for any graph G on 9 vertices, $\gamma = \gamma' = 3$ if and only if $G = K_3 \times K_3$. Arumugam and Paulraj Joseph [7] studied the class of graphs for which connected domination number and domination number are equal.

In this paper we initiate a study of graphs in which some of the edge domination parameters are equal. We characterize connected graphs for which $\gamma' = \gamma'_c$.

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2. Main Results

Lemma 2.1. Let G be a connected graph with $\gamma' = \gamma_c' = n$. Then for every minimum connected edge dominating set S of G, the edge induced sub graph $\langle S \rangle$ is isomorphic to $K_{1,n}$.

Proof. Let S be any minimum connected edge dominating set of G. If $\langle S \rangle$ contains two vertices u, v of degree at least two, then $S/\{e\}$ where e is any non – pendant edge of S incident with u forms an edge dominating set of G so that $\gamma' < \gamma_c'$ which is a contradiction. Hence at most one vertex of $\langle S \rangle$ has degree greater than 1. Thus $\langle S \rangle = K_{1,n}$.

Corollary 2.2. Let G be a connected graph. If $\gamma' = \gamma_c'$ then diam (G) ≤ 4.

Proof. Suppose $\gamma' = \gamma_c' = n$. Then there exists a star $K_{1,n}$ in G such that all the e dges of G are incident with the vertices of $K_{1,n}$ and for every edge e_i of $K_{1,n}$, there exists an edge x_i of G such that x_i is adjacent to e_i but not adjacent to any other edge of $K_{1,n}$. Hence diam (G) ≤ 4.

The converse of Corrollary 2.2 is not true. For example, for the graph G given in Figure 2.1, diam (G) = 4, $\gamma' = 4$ and $\gamma_c' = 6$.

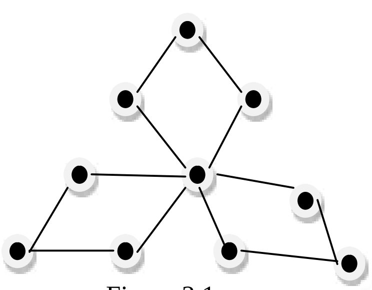

Figure 2.1

Corrollary 2.3. For any tree T, $\gamma' = \gamma_c'$ if and only if diam (G) ≤ 4.

Proof. If $\gamma' = \gamma_c'$, then diam (G) ≤ 4.

Consider let diam (G) ≤ 4. If diam(T) = 4 then , $\gamma' = \gamma_c' = q - e'$ where e' is the number of pendant edges of T. If diam(T) < 4, then $\gamma' = \gamma_c' = 1$.

Definition 2.4. Let $\mathcal{P}$ be a property that , $\gamma' = \gamma_c' = n$. A connected graph G is said to be $\mathcal{P}$ – critical if G satisfies $\mathcal{P}$ and no proper connected subgraph H of G satisfies $\mathcal{P}$.

Lemma 2.5 Let $\mathcal{P}$ denote the property that $\gamma' = \gamma_c' = n \geq 3$. A connected graph G is said to be $\mathcal{P}$ – critical if and only if G is isomorphic to $S(K_{1,n})$

Proof. Let G be a connected graph which is $\mathcal{P}$ – critical so that $\gamma' = \gamma_c' = n$. Let S be any minimum connected edge dominating set of G. By Lemma 2.1, $\langle S \rangle = K_{1,n}$. Le: $V(\langle S \rangle) = \{u, u_1, u_2, \dots, u_n\}$ with $deg_{\langle S \rangle} u = n$. Let e = uu_i $(1 \leq i \leq n)$. Since S is a minimum connected edge dominating set of G, for edge e_i, there exists an edge x_i of G such that x_i is adjacent to e_i but not adjacent to $e_j, j \neq i$. Further every edge of G has at least one of its ends in $\{u, u_1, u_2, \dots, u_n\}$. Since $\gamma'(G) = \gamma_c'(G) = n$, we may assume without loss of generality that at least n – 1 of the edges $x_1, x_2, \dots, x_n$ are independent. Hence the subgraph H induced by the edges $e_1, e_2, \dots, e_n, x_1, x_2, \dots$, x_n is isomorphic to $S(K_{1,n})$. Clearly $\gamma'(H) = \gamma_c'(H) = n$ and since G is $\mathcal{P}$ – critical, it follows that G = H. The converse is obvious.

Remark 2.6 Let G be a connected graph. Then $\gamma' = \gamma_c' = 1$ if and only if there exists an edge $e = uv$ such that every edge of edge of G is incident with u or v

Theorem 2.7 Let G be a connected graph. Then $\gamma' = \gamma_c' = 2$ if and only if G contains a subgraph H isomorphic to $K_{1,2}$ and every edge e = uv in E(G) \ E(H) has the following properties.

(i) At least one of u, v is in V($K_{1,2}$) $\subseteq V(H)$

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(ii) If u, v are pendant vertices of $K_{1,2}$ in H, then the center of $K_{1,2}$ has degree at least three in G.

Proof. Suppose $\gamma' = \gamma'_c = 2$. Let S be any minimum connected edge dominating set of G. Then by Lemma 2.1,

$H = \langle S \rangle = K_{1,2}$. Let $V(K_{1,2}) = \{u, u_1, u_2\}$ with $deg_H u = 2$ Since $E(K_{1,2})$ is a minimum connected edge dominating set of G, every edge of E(G) \ E(H) has at least one of its ends in $V(K_{1,2})$. If $u_1 and\ u_2$ are adjacent in G, then $deg_G u = 2$, then $\{u_1 u_2\}$ is an edge dominating set of g so that $\gamma'(G) = 1 < \gamma'_c(G)$G contains a subgraph H isomorphic to $K_{1,2}$ and every edge e = uv in E(G) \ E(H) has the following properties.

which is a contradiction. Therefore if $u_1 and\ u_2$ are adjacent in G, then $deg_G u \geq 3$. Conversely suppose that

G contains a subgraph H isomorphic to $K_{1,2}$ and every edge e = uv in E(G) \ E(H) has the following properties (i) and (ii) mentioned in Theorem 2.5. Then $K_{1,2}$ is a subraph G and $E(K_{1,2})$ is a connected edge dominating set of G so that $\gamma'_c \leq 2$. Hence $\gamma' \leq \gamma'_c \leq 2$. Since for edge e of G, there exists an edge x of G such that x and e are not adjacent, $\gamma' \geq 2$. Hence $\gamma' = \gamma'_c = 2$.

Theorem 2.8 Let $G_1 = S(K_{1,n})$ with $V(G_1) = \{w, u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ and $E(G_1) = \{wu_i / 1 \leq i \leq n\} \cup \{u_i v_i / 1 \leq i \leq n\}$. Then for a connected graph G, $\gamma' = \gamma'_c = n \geq 3$ if and only if G satisfies the following conditions.

(1) G contains a subgraph H isomorphic to G_1

(2) Every edge of E(G) \ E(H) has exactly one of its ends in $\{w, u_1, u_2, \dots, u_n\}$

(3) If $u_i v_j \in E(G)$ ($i \neq j$), then $u_j v_i \in E(G)$ or $u_j v_i \in E(G)$ or $u_j v_k$ for all $k \neq i$ or $u_j w \in E(G)$ for some w in V(G) \ V(H)

(4) If $u_i v_j$, $u_j v_i$ and $v_i v_k$ are in G $(i \neq j \neq k)$ then $u_j v_k \in E(G)$or $u_j v_m \in E(G)$ for all $m \neq k$ or $u_j w \in E(G)$ for some w in $V(G)/V(H)$

Proof. Suppose $\gamma' = \gamma'_c = n \geq 3$. Let S be any minimum connected edge dominating set of G. By Lemma 2.1, $\langle S \rangle = K_{1,n}$. Let $V(\langle S \rangle) = \{w, u_1, u_2, \dots, u_n\}$ and$deg_{\langle S \rangle} w = n$. Let $e_i = wu_i$. Then as in Lemma 2.5, for each edge e_i, we can choose an edge x_i such that x_i is adjacent to $e_i$but not adjacent to any other edge of S and the subgraph H induced by e_i's and x_i's is $\mathcal{P}$ - critical. Hence H is isomorphic to $S(K_{1,n})$. Since S is minimum connected edge dominating set of G, every edge of $E(G)/E(H)$ has at least one end in $V(K_{1,n})$. If there exists vertices u_i, u_j such that $u_i u_j \in E(G)$, then $(E(K_{1,n}) \backslash \{wu_i, wu_j\}) \cup \{u_i u_j\}$ is an edge dominating set of G so that $\gamma'(G) < \gamma'_c(G)$ which is a contraction. Hence every edge of $E(G)/E(H)$ has exactly one of its ends in $\{w, u_1, u_2, \dots, u_n\}$. Now let $u_i v_j \in E(G)$. Suppose $u_j v_i \notin E(G)$ and $u_j w \notin E(G)$ for every vertex w in V(G) \ V(H). We claim that $u_j v_k \in E(G)$ for all $\neq i$. Suppose $u_j v_k \notin E(G)$ for some $k \neq i$. Without loss of generality we assume that $1 \leq i < j < k \leq n$.Then

$$S_1 = \{wu_k, u_i v_j, u_1 v_1, u_2 v_2, \dots, u_{i-1} v_{i-1}, u_{i+1} v_{i+1}, \dots, u_{j-1} v_{i-1}, u_{j+1} v_{j+1}, \dots, u_{k-1} v_{k-1}, u_{k+1} v_{k+1}, \dots, u_n v_n\}$$

is an edge dominating set of G of cardinality n – 1 so that $\gamma'(G) < \gamma'_c(G)$ which is a contradiction. Hence $u_j v_k \in E(G)$ for all $k \neq i$. Thus (3) is proved.

Now let $u_i v_j, u_j v_i$ and $v_i u_k$ be in E(G) ($I \neq j \neq k$). Suppose $u_j v_k \notin E(G)$ and $u_j w \notin E(G)$ for every vertex w in $V(G)/V(H)$. We claim that

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$u_j v_m \in E(G)$ for all $m \neq k$. Suppose $u_j v_m \notin E(G)$ for some $m \neq k$.. Without loss of generality we assume that $1 \leq k < i < j < m \leq n$. Then

$$S_2 = \left\{ \begin{matrix} wu_m, u_i v_j, u_k v_i, u_1 v_1, u_2 v_2, \dots, \\ u_{k-1} v_{k-1}, u_{k+1} v_{k+1}, \dots, \\ u_{i-1} v_{i-1}, u_{i+1} v_{i+1}, u_{j-1} j, u_{j+1} v_{j+1}, \dots, \\ u_{m-1} v_{m-1}, u_{m+1} v_{m+1}, \dots u_n v_n \end{matrix} \right\}$$ is an edge dominating set of cardinality n – 1 so that $\gamma'(G) < \gamma_c'(G)$ which is a contradiction. Hence $u_j v_m \in E(G)$ for all m≠ k. Thus (4) is proved.

Conversely let us assume that G satisfies (1), (2), (3), and (4) mentioned in the hypothesis. We claim that $\gamma' = \gamma_c' = n$. Clearly $\{wu_1, wu_2, \dots, wu_n\}$ is a connected edge dominating set of G so that $\gamma' \leq \gamma_c' \leq n$. Now let D be any minimum independent edge dominating set of G so that $|D| = \gamma' = \gamma_i'$.

Suppose D contains no edge incident with w. Since $\{u_1, u_2, \dots, u_n\}$ is independent in G, D must contains at least n edges for dominating the edges $wu_1, wu_2, \dots, wu_n$ so that $|D| \geq n$.

Suppose D contains an edge incident with w, say wu_1. If D contains an edge incident with u_i, for each i, $2 \leq i \leq n$, then $|D| \geq n$.

Suppose D does not contain any edge incident with u_2 and if H is isomorphic to C_1. Since D contains no edge incident with u_2, $u_2 w \notin E(G)$ for any vertex w in V(G) \ V(H). Now for dominating $u_2 v_2$, D must contain an edge incident with u_2, say $v_2 u_3$. Then by (4), $u_2 v_3 \in E(G)$ or $u_2 v_k \in E(G)$ for all k $\neq 3$. Suppose $u_2 v_k \in E(G)$ for all k $\neq 3$. Since $\{v_1, v_2, \dots, v_n\}$ is independent in G for dominating edges $u_2 v_1, u_2 v_2, \dots, u_2 v_n$, D must contain n – 2 edges so that $|D| \geq 2 + n - 2 = n$. If $u_2 v_3 \in E(G)$, then for dominating $u_2 v_3$ D must contain an edge incident with v_3, say $v_3 u_4$. Again by (4), $u_2 v_4 \in E(G)$ or $u_2 v_k \in E(G)$ for all k $\neq 4$. Continuing this process, we get $|D| \geq n$. Thus $\gamma'(G) \geq n$ so that $\gamma' = \gamma_c' = n$.

Corollary 2.9 Let G be a connected unicyclic graph with unique cycle C. Then $\gamma' = \gamma_c' = n$ if and only if the following holds.

1. C = C_3 or C_4
2. Every vertex not on C has degree 1 or 2, all vertices of degree 2 not on C are adjacent to the same vertex u of C and the distance between any pendant vertex and C is 1 or 2
3. If $C = C_4$ and every vertex not on C_4 is a pendant vertex , then at least one vertex of C_4 is of degree 2.
4. If there exists a vertex of degree 2 not on C, then at least one vertex v of C has degree two and when $C = C_4$, v is non adjacent to u

Proof Let G be a connected unicyclic graph with cycle C and $\gamma' = \gamma_c'$. If $\gamma' = \gamma_c' = 1$ then $C = C_3$, every vertex not C_3 has degree 2. Suppose $\gamma' = \gamma_c' = n \geq 2$. Let S be a minimum connected edge dominating set of G. Then by Lemma 2.1, $\langle S \rangle = K_{1,n}$. Let $V(\langle S \rangle) = \{w, u_1, u_2, \dots, u_n\}$ and $deg_{\langle S \rangle} w$= n. Since every edge of G has at least one end in $\langle S \rangle$, it follows *that* $C = C_3$ *or* C_4. *If* $n = 2$ *and* $u_1 u_2 \in E(G)$, *then by Theorem 2.7 it follows that* $C = C_3$, *every vertex not on* C_3 *is a* pendant vertex and every vertex on C_3 has degree at least 3. If $n \geq 3$ and there exists a vertex $v_1 \notin V(\langle S \rangle)$ such that $u_1 v_1, u_2 v_1 \in E(G)$ then $C = C_4$ and the result follows from Theorems 2.7 and 2.9. The converse is obvious.

Theorem 2.12 For a connected cubic graph G, $\gamma' = \gamma_c'$ if and only if G is isomorphic to K_4 or the graph given in Figure 2.2

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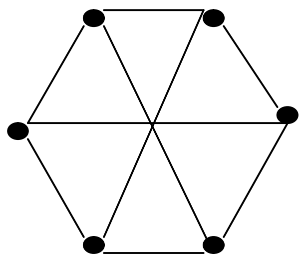

Figure 2.2

Proof Suppose $\gamma'=\gamma_c'=n$. Let S be any minimum connected edge dominating set of G. By Lemma 2.1, $\langle S\rangle = K_{1,n}$. Since G is a cubic graph, it follows that n = 2 or 3

Case (i) $\gamma'=\gamma_c'=2$.

Since $\gamma' \leq \frac{p}{2}$ $\gamma' \geq \frac{q}{\Delta'+1}$ and $\Delta' = 4$, we have $p \geq 4$ *and* $q \leq 10$. Since every edge of G is incident with the vertices of $\langle S\rangle = K_{1,2}$ and G is cubic, it follows that $q \leq 7$. Hence p = 4 and G is isomorphic to K_4.

Case (ii) $\gamma'=\gamma_c'=3$.

In this case $p \geq 6$ *and* $q \leq 15$. Since every edge of G is incident with the vertices of $\langle S\rangle = K_{1,3}$ and G is cubic, it follows that $q \leq 9$. Hence p = 6. Now by theorem 2.9, G contains a subgraph H isomorphic to the graph obtained by identifying two pendant vertices of $S(K_{1,3})$ so that G is isomorphic to the graph given in Figure 2.3.

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