Analysis of Fuzzy Queues with Bulk Service

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***Abstract*–This paper proposes a general procedure to construct the membership functions of the performance measures in fuzzy queuing systems with bulk service when the interarrival time and service time are fuzzy numbers. The basic idea is to reduce a fuzzy queue into a family of crisp queues[11] by applying the α-cut approach. A Mathematical investigation has been made of a simple queueing process in which customers arrive at random, form a single queue in order of arrival, and are served in batches, the size of each batch having a fixed maximum. Triangular fuzzy numbers are used to demonstrate the validity of the proposal. Numerical examples are illustrated successfully.**

***Keywords*–Fuzzy Number, Queueing Theory, Parametric Programming, Membership Functions.**

I. INTRODUCTION

The problem of fuzzy queues has been analysed by prade [9] and Li and lee [6] through the use of extension principle [10]. However, these approaches are fairly complicated and are generally unsuited for computational purposes. N.T.J. Bailey[1] has introduced the concept of bulk queues. He has considered a situation where the service can be effected in a batch up to C-customers, that is, all waiting customers up to fixed capacity 'C' are taken for service in a batch. It is a more general type queueing problem which can be specialized to yield the single service cases. D.G. Kendall [4] has discussed and developed the concept of imbedded Markov chain. This type of bulk service is called "general bulk service" and has been analyzed by Neuts[8]. The fuzzy queues namely M/F/1, F/M/1,F/F/1, and FM/FM/1 are exemplified in [2]. It has been further investigated by Gross and Harris [3] and others.

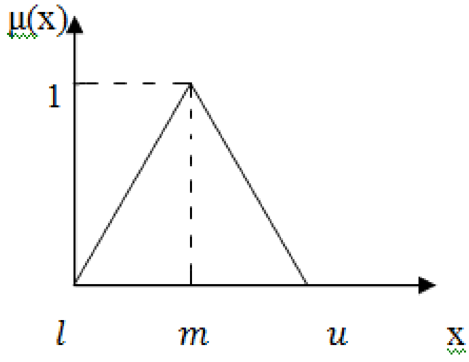
Only a triangular fuzzy number is used to represent the fuzziness. We feel that a triangular fuzzy number is an ideal compromise between complexity and over simplification.

II. TRIANGULAR FUZZY NUMBER

We define a fuzzy number M on R to be a triangular fuzzy number if its membership function $\mu_M(x): R \to [0,1]$ is defined by

$$\mu_M(x) = \begin{cases} \dfrac{x-l}{m-l}, & \text{for } l \le x \le m \\ \dfrac{x-u}{m-u}, & \text{for } m \le x \le u \\ 0, & \text{otherwise} \end{cases}$$

where $l \le m \le u$, l and u stand for the lower and upper value of the support of M respectively and m for the modal value. The triangular fuzzy number can be denoted by (l,m,u). The support of M is the set of elements $\{x \in R / l \prec x \prec u\}$, when $l = m = u$, it is a non-fuzzy number by convention. Here the vertical line shows the membership function.



III. FUZZY SET THEORY [5]

Definition.3.1. Let X be a classical set of objects, called the universe , whose generic elements are denoted by x. Membership is a classical subset A of X is often viewed as a characteristic function $\mu_{\tilde{A}}$ from X to [0,1] such that

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 & \text{when } x \in \tilde{A} \\ 0 & \text{when } x \notin \tilde{A} \end{cases}$$

[0,1] is called a valuation set

Definition.3.2. The support of a fuzzy set A is the ordinary subset of X, such that SuppA = $\{x \in X / \mu_{\tilde{A}}(x) \succ 0\}$

Definition.3.3. The height of Ã is $h(\tilde{A}) = \sup_{x \in X} \mu_{\tilde{A}}(x)$, ie, the least upper bound of $\mu_{\tilde{A}}(x)$.

Definition.3.4. Ã is said to be normalized iff there exist x ∈ X, such that $\mu_{\tilde{A}}(x) = 1$; this definition implies that $h(\tilde{A}) = 1$. Otherwise Ã is said to be subnormal.

Level set.3.5. The set of elements that belong to the fuzzy set Ã at least to the degree 'α' is called the α -level set

$A_\alpha = \{ x \in X / \mu_{\tilde{A}}(x) \geq \alpha \}$ and

$A_\alpha^+ = \{ x \in X / \mu_{\tilde{A}}(x) > \alpha \}$

is called "strong α - level set" or "strong α - cut"

Definition.3.6. If a fuzzy set Ã is defined on X, for any $\alpha \in [0,1]$ the α - cuts of the fuzzy set Ã is represented by $\tilde{A}_\alpha = \{x \in X / \mu_{\tilde{A}}(x) \geq \alpha\} = \{\ell_{\tilde{A}}(\alpha), u_{\tilde{A}}(\alpha)\}$, where $\ell_{\tilde{A}}(\alpha)$ and $u_{\tilde{A}}(\alpha)$ represent the lower bound and upper bound of α - cut of Ã respectively.

IV. Problem Formulation

Formulation of parametric programming problem for bulk service queueing model.

Consider a single - server markovian queue with bulk service. The interarrival time $\tilde{A}$ and service time $\tilde{S}$ are approximately known and are represented by the following fuzzy sets.

$\tilde{A} = \{(a, \mu_{\tilde{A}}(a)) / a \in X\}$ …(4.1)

$\tilde{S} = \{(S, \mu_{\tilde{S}}(s)) / S \in Y\}$ ….(4.2)

where X and Y are crisp universal sets of the interarrival time and bulk service time and $\mu_{\tilde{A}}(a)$, $\mu_{\tilde{S}}(s)$ are the respective membership functions.

The α - cut of Ã and $\tilde{S}$ are

$A(\alpha) = \{a \in X / \mu_{\tilde{A}}(a) \geq \alpha\}$ ….(4.3)

$S(\alpha) = \{s \in Y / \mu_{\tilde{S}}(s) \geq \alpha\}$ …(4.4)

where $0 < \alpha \leq 1$. Both A(α) and S(α) are the crisp sets. Using α -cut, the interarrival times and bulk service times can be represented by different levels of confidence intervals. Hence a fuzzy queue can be reduced to a family of crisp queues with different α - level cuts $\{A(\alpha) / 0 < \alpha \leq 1\}$ and $\{S(\alpha) / 0 < \alpha \leq 1\}$.These two sets represent sets of movable boundaries and they form nested structure for expressing the relationship between the crisp sets and fuzzy sets [5].

Let the confidence intervals of the fuzzy sets $\tilde{A}$ and $\tilde{S}$ be $[l_{A(\alpha)}, u_{A(\alpha)}]$ and $[l_{S(\alpha)}, u_{S(\alpha)}]$ respectively. Since both the interarrival time $\tilde{A}$ and bulk service time $\tilde{S}$ are fuzzy numbers, using zadeh's extension principle [5,11], the membership function of the performance measure $P(\tilde{A}, \tilde{S})$ is defined [6] as

$$\mu_{p(\tilde{A},\tilde{S})}(z) = \sup_{a \in X,\, S \in Y} \min \{\mu_{\tilde{A}}(a), \mu_{\tilde{S}}(s) / z = P(a,s)\} \quad \text{……..(4.5)}$$

Construction of the membership function $\mu_{p(\tilde{A},\tilde{S})}(Z)$ is equivalent to say that the derivation of α - cuts of $\mu_{p(\tilde{A},\tilde{S})}(Z)$. From equation (4.5) the equation $\mu_{p(\tilde{A},\tilde{S})}(Z) = \alpha$ is true only when either

$\mu_{\tilde{A}}(a) = \alpha$, $\mu_{\tilde{S}}(S) \geq \alpha$ (or) $\mu_{\tilde{A}}(a) \geq \alpha$, $\mu_{\tilde{S}}(S) = \alpha$ is true.

The parametric programming problems have the following form

$$\left.\begin{array}{l} \ell_{P(\alpha)} = \min P(a,s) \text{ such that} \\ \ell_{A(\alpha)} \leq a \leq u_{A(\alpha)} \\ \ell_{S(\alpha)} \leq s \leq u_{S(\alpha)} \end{array}\right\} \quad \text{…(4.6)}$$

and

$$\left.\begin{array}{l} u_{P(\alpha)} = \max P(a,s) \text{ such that} \\ \ell_{A(\alpha)} \leq a \leq u_{A(\alpha)} \\ \ell_{S(\alpha)} \leq s \leq u_{S(\alpha)} \end{array}\right\} \quad \text{….(4.7)}$$

If both $\ell_{P(\alpha)}$ and $u_{P(\alpha)}$ are invertible with respect to α, then the left shape function $L(z) = \ell^{-1}_{P(\alpha)}$ and the right shape function $R(z) = u^{-1}_{p(\alpha)}$ can be obtained from which the membership function $\mu_{p(\tilde{A},\tilde{S})}(z)$ is constructed as

$$\mu_{p(\tilde{A},\tilde{S})}(z) = \begin{cases} L(z) & \text{for } Z_1 \leq Z \leq Z_2 \\ R(z) & \text{for } Z_2 \leq Z \leq Z_3 \\ 0 & \text{otherwise} \end{cases} \quad \text{……….(4.8)}$$

where $Z_1 \leq Z_2 \leq Z_3$, $L(Z_1) = R(Z_3) = 0$, and $L(Z_2) = R(Z_2) = 1$.

By traditional queuing theory, to find the measures of effectiveness, the steady - state probability distribution for the system size allows to find the expected number of customers in the system and expected number of customers in the queues at steady- state, Let N be the random variable representing the "number of customers in the system in steady-state and L be its expected value then

$$L = E(N) = \sum_{n=0}^{\infty} n P_n$$

$$= (1-\rho) \sum n\rho^n \qquad \text{.......(4.9)}$$

$$= (1-\rho)\rho \sum_{n=1}^{\infty} n\rho^{n-1}$$

$$L = \frac{\rho}{1-\rho} \text{, and since } \rho = \lambda/\mu$$

$$L = \frac{\lambda}{\mu-\lambda} \qquad \text{......(4.10)}$$

Let N_q be the random variable representing the number in the queue at steady- state, and L_q be its expected value, then we have

$$L_q = \sum_{n=1}^{\infty}(n-1)P_n = \sum_{n=1}^{\infty} n P_n - \sum_{n=1}^{\infty} P_n$$

$$= L - (1-P_o)$$

$$= \frac{\rho}{1-\rho} - \rho \qquad (\because \rho = \frac{\lambda}{\mu})$$

$$= \frac{\rho^2}{1-\rho}$$

$$L_q = \frac{\lambda^2}{\mu(\mu-\lambda)} \qquad \text{.........(4.11)}$$

Using little's formula, $L = \lambda W$, and $L_q = \lambda W_q$

$$\text{Hence } W = \frac{L}{\lambda} = \frac{\rho}{\lambda(1-\rho)} = \frac{1}{\mu-\lambda} \quad \text{.... (4.12)}$$

$$W_q = \frac{L_q}{\lambda} = \frac{\rho}{\mu(1-\rho)}, \qquad (\because \rho = \frac{\lambda}{\mu})$$

$$W_q = \frac{\rho}{\mu-\lambda} = \frac{\lambda}{\mu(\mu-\lambda)} \qquad \text{......(4.13)}$$

This procedure has been applied to find the membership function of the performance measures for $FM/FM^{(y)}/1$ (bulk service queuing) model. Little's formula is verified for considering the respective intervals in $[\ell_{L(\alpha)}, u_{L(\alpha)}]$, $[\ell_{Lq(\alpha)}, u_{Lq(\alpha)}]$, $[\ell_{W(\alpha)}, u_{W(\alpha)}]$ and $[\ell_{Wq(\alpha)}, u_{Wq(\alpha)}]$ using the operations defined on the intervals.

V. NUMERICAL EXAMPLE

Consider an $FM/FM^y/1$ queuing model, where both arrival rate and bulk service rate are triangular fuzzy numbers, represented by $\widetilde{A}$ = [2,3,4] and $\widetilde{S}$ = [16,17,18]. The α - cut of the membership functions $\mu_{\widetilde{A}}$ and $\mu_{\widetilde{S}}$ are respectively, [(2+α), (4-α)] and [(16+α), (18-α)]. To derive the membership function for $\widetilde{L}$ the following parametric programming problem, (4.6) and (4.7) are considered.

$$\ell_{L(\alpha)} = \min \left(\frac{x}{y-x}\right) \text{ such that } 2+\alpha \le x \le 4-\alpha,\ 16+\alpha \le y \le 18-\alpha \qquad \text{......(5.1)}$$

and

$$u_{L(\alpha)} = \max \left(\frac{x}{y-x}\right) \text{ such that } 2+\alpha \le x \le 4-\alpha,\ 16+\alpha \le y \le 18-\alpha \qquad \text{........(5.2)}$$

$\ell_{L(\alpha)}$ is found when x approaches its lower bound and y approaches its upper bound. Consequently the optimal solution for (5.1) is

$$\ell_{L(\alpha)} = \frac{2+\alpha}{16-2\alpha} \text{.........(5.3)}$$

also $u_{L(\alpha)}$ is found when x approaches its upper bound and y reaches its lower bound. In this case, the optimal solution for (5.2) is

$$u_{L(\alpha)} = \frac{4-\alpha}{12+2\alpha} \qquad \text{............(5.4)}$$

The above functions in equations (5.3) and (5.4) are invertible. Applying the inverse transformation, the performance function $\mu_{\widetilde{L}}(z)$ of $\widetilde{L}$ is obtained and .The membership function $\mu_{\widetilde{L}}(z)$ is obtained as

$$\mu_{\widetilde{L}}(z) = \begin{cases} \dfrac{16Z-2}{2Z+1} & \text{for } .125 \le Z \le .2143 \\ \dfrac{4-12Z}{2Z+1} & \text{for } .2143 \le Z \le .3333.. \\ 0 & \text{otherwise} \end{cases} \qquad \text{..........(5.5)}$$

which is the performance function for $\widetilde{L}$ in $FM/FM^y/1$ queuing model.

The performance functions of (i) $\widetilde{L}_q$-average queue length (ii) $\widetilde{W}$-average waiting time in the system and (iii) $\widetilde{W}_q$-average waiting time in the queue, are derived from the respective parametric programs. They differ only in the objective functions and are listed below.

(i) $$\ell_{L_q(\alpha)} = \min \left(\frac{x^2}{y(y-x)}\right) \qquad \text{........(5.6)}$$

and

$$u_{L_q(\alpha)} = \max \left(\frac{x^2}{y(y-x)}\right) \qquad \text{......(5.7)}$$

(ii) $$\ell_{W(\alpha)} = \min \left(\frac{1}{y-x}\right) \qquad \text{.........(5.8)}$$

and

$$u_{W(\alpha)} = \max \left(\frac{1}{y-x}\right) \qquad \text{....(5.9)}$$

(iii) $\ell_{W_q(\alpha)} = \min\left(\frac{x}{y(y-x)}\right)$ …….(5.10)

and

$u_{W_q(\alpha)} = \max\left(\frac{x}{y(y-x)}\right)$. ……(5.11)

The objective functions listed from equation (5.6) to (5.11) together with the constraints given by the equations (5.1) and (5.2) yield the following results.

$$\ell_{L_q(\alpha)} = \frac{4+4\alpha+\alpha^2}{(288-52\alpha+2\alpha^2)} \quad ….(5.12)$$

$$u_{L_q(\alpha)} = \frac{16-8\alpha+\alpha^2}{(192+44\alpha+2\alpha^2)} \quad ……..(5.13)$$

$$\mu_{\tilde{L}_q}(z) = \begin{cases} \dfrac{(52Z+4)-\sqrt{400Z^2+1600Z}}{2(2Z-1)} & \text{for } .01389 \le Z \le .037815 \\ \dfrac{-(44Z+8)+\sqrt{400Z^2+1600Z}}{2(2Z-1)} & \text{for } .037815 \le Z \le .08333 \\ 0 & \text{otherwise} \end{cases}$$

……(5.14)

$$\ell_{W(\alpha)} = \frac{1}{16-2\alpha} \quad …(5.15)$$

$$u_{W(\alpha)} = \frac{1}{12+2\alpha} \quad ……(5.16)$$

$$\mu_{\tilde{W}}(z) = \begin{cases} \dfrac{16Z-1}{2Z} & \text{for } .0625 \le Z \le .07143 \\ \dfrac{1-12Z}{2Z} & \text{For } .07143 \le Z \le .08333 \\ 0 & \text{otherwise} \end{cases}$$

……(5.17)

$$\ell_{W_q(\alpha)} = \frac{2+\alpha}{288-52\alpha+2\alpha^2} \quad ………..(5.18)$$

$$u_{W_q(\alpha)} = \frac{4-\alpha}{192+44\alpha+2\alpha^2} \quad ……..(5.19)$$

$$\mu_{\tilde{W}_q}(z) = \begin{cases} \dfrac{(52Z+1)-\sqrt{400Z^2+120Z+1}}{4Z} & \text{for } .006944 \le Z \le .0126 \\ \dfrac{-(44Z+1)+\sqrt{400Z^2+120Z+1}}{4Z} & \text{for } .0126 \le Z \le .02083 \\ 0 & \text{otherwise} \end{cases}$$

….(5.20)

Table.1. gives the α - cuts of arrival rate and bulk service rate for FM/FM^y/1 queuing model for the above data, and Figures 1 to 4 gives the graphs for the corresponding membership function solutions.

TABLE 1
α -CUTS OF ARRIVAL RATE AND BULK SERVICE RATE USING TRIANGULAR FUZZY NUMBERS

α	$\ell_{x(\alpha)}$	$U_{x(\alpha)}$	$\ell_{y(\alpha)}$	$U_{y(\alpha)}$	$\ell_{L(\alpha)}$	$U_{L(\alpha)}$	$\ell_{Lq(\alpha)}$	$U_{Lq(\alpha)}$	$\ell_{W(\alpha)}$	$U_{W(\alpha)}$	$\ell_{Wq(\alpha)}$	$U_{Wq(\alpha)}$
0	2	4	16	18	0.1250	0.3333	0.0139	0.0833	0.0625	0.0833	0.0069	0.0208
0.1	2.1	3.9	16.1	17.9	0.1329	0.3197	0.0156	0.0774	0.0633	0.0820	0.0074	0.0199
0.2	2.2	3.8	16.2	17.8	0.1410	0.3065	0.0174	0.0719	0.0641	0.0806	0.0079	0.0189
0.3	2.3	3.7	16.3	17.7	0.1494	0.2937	0.0194	0.0667	0.0649	0.0794	0.0084	0.0180
0.4	2.4	3.6	16.4	17.6	0.1579	0.2813	0.0215	0.0617	0.0658	0.0781	0.0090	0.0171
0.5	2.5	3.5	16.5	17.5	0.1667	0.2692	0.0238	0.0571	0.0667	0.0769	0.0095	0.0163
0.6	2.6	3.4	16.6	17.4	0.1757	0.2576	0.0263	0.0528	0.0676	0.0758	0.0101	0.0155
0.7	2.7	3.3	16.7	17.3	0.1849	0.2463	0.0289	0.0487	0.0685	0.0746	0.0107	0.0147
0.8	2.8	3.2	16.8	17.2	0.1944	0.2353	0.0317	0.0448	0.0694	0.0735	0.0113	0.0140
0.9	2.9	3.1	16.9	17.1	0.2042	0.2246	0.0346	0.0412	0.0704	0.0725	0.0119	0.0133
1.0	3.0	3.0	17.0	17.0	0.2143	0.2143	0.0378	0.0378	0.0714	0.0714	0.0126	0.0126

VI. Expected Number Of Customers In The System

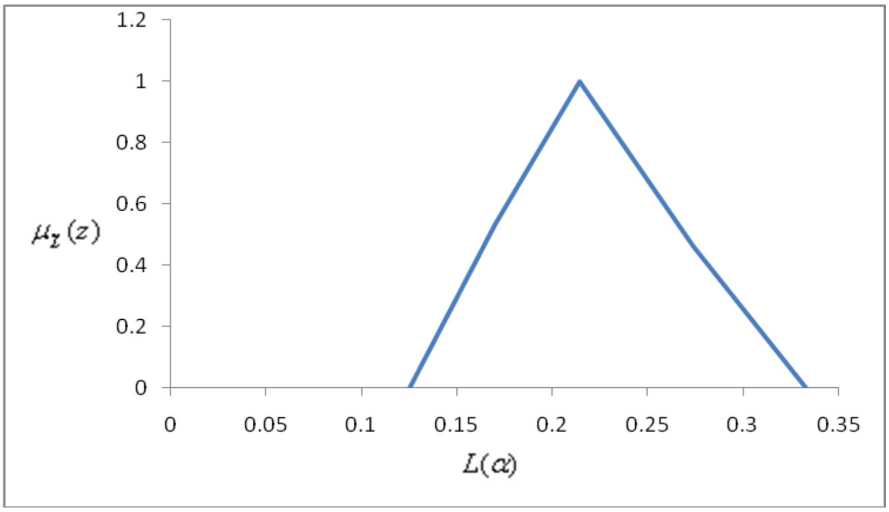


Fig. 1 Expected Number Of Cutomers In The Queue

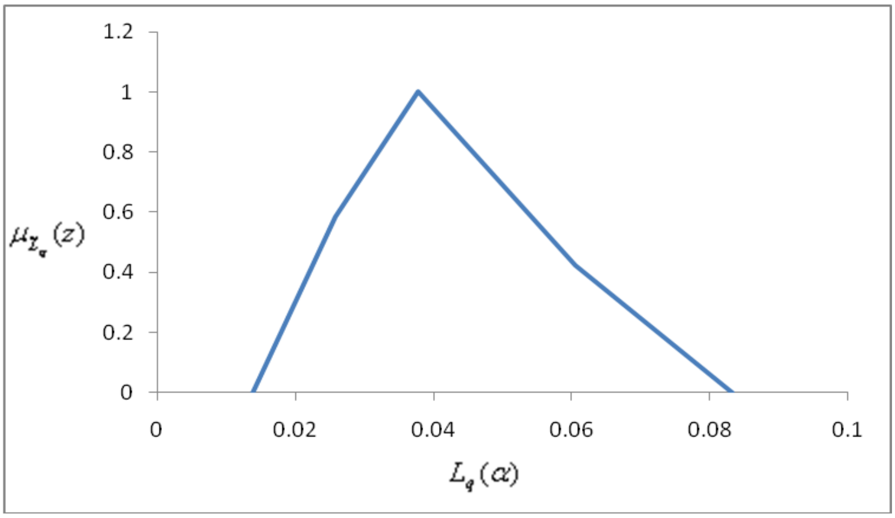


Fig. 2 Average Waiting Time In The System

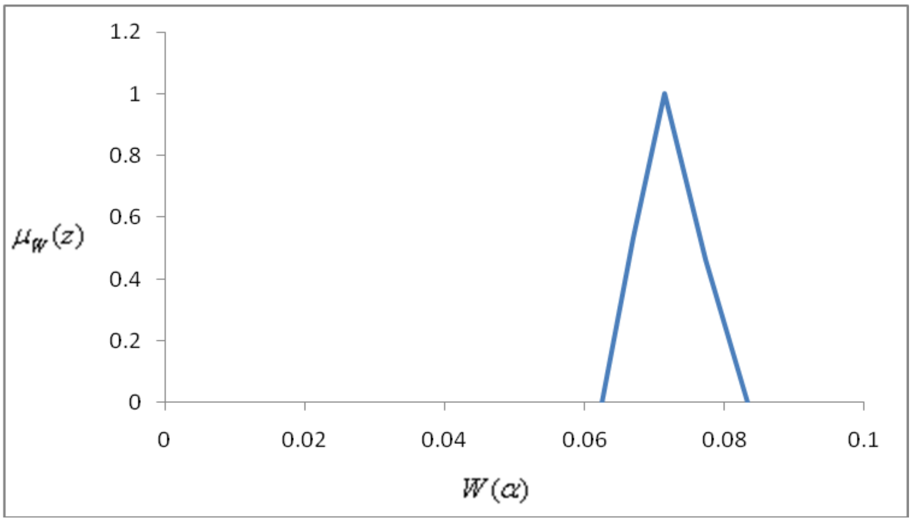


Fig. 3 Average Waiting Time In The Queue

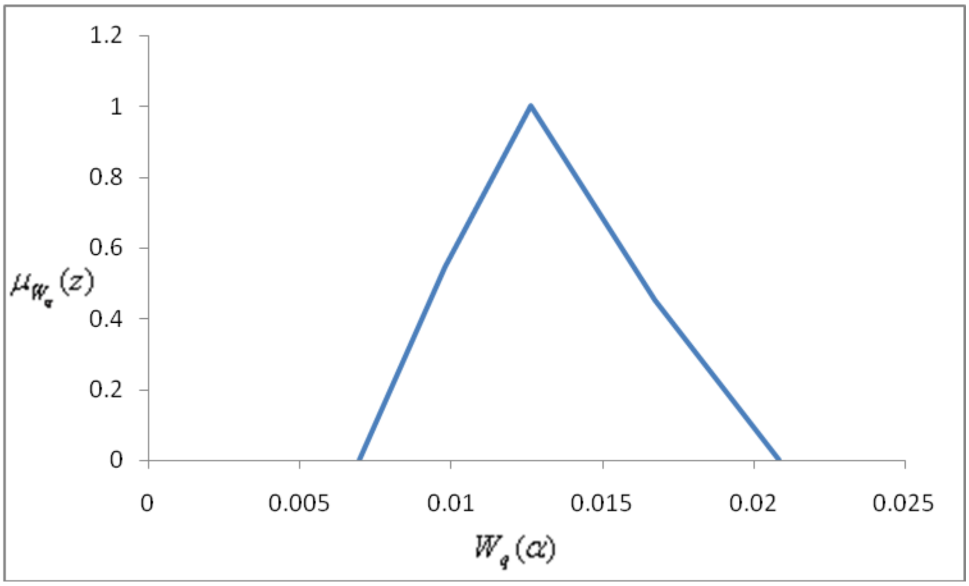


Fig. 4

VII. Conclusion

Fuzzy set theory has been applied to some classical queueing systems to provide wider applications in some previous studies [5,6]. When the interarrival time and bulk service time are fuzzy variables, according to zadeh's extension principle[9], the performance measures such as the average system length, the average waiting time, etc. will be fuzzy as well. This paper applies the concept of α-cut to reduce a fuzzy queue into a family of crisp queues which can be described by a pair of parametric programs to find the α-cuts of the membership functions of the performance measures. Although this is not a very precise procedure, it does give a very useful quick practical guide to the number of patients who should be dealt with each week in any given speciality, if seriously lengthy periods of waiting are to be avoided. For a more searching analysis the detailed results of this paper can be applied.

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