New Bounds for the Zeros of a Polynomial

M. H. Gulzar[1], G. M. Wani[2]
1. Department of Mathematics, University of Kashmir, Srinagar ,India
2. Department of Mathematics, Govt. Degree College Bemina, Srinagar, India

1. gulzarmh@gmail.com

2. gmwani6@gmail.com

Abstract— **In this paper we give new bounds for the zeros of a polynomial satisfying certain conditions.**
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I. Introduction

The following result known as the Enestrom-Kakeya Theorem is an elegant result in the theory of the distribution of zeros of polynomials:

Theorem A: If the coefficients of the polynomial $P(z)=\sum_{j=0}^{n}a_jz^j$ satisfy $0\le a_0\le a_1\le ......\le a_{n-1}\le a_n$, then all the zeros of P(z) lie in the closed disk $|z|\le 1$.

In the literature ([2], [4]-[6], [8]-[12]) there exist several generalizations of this result. Aziz and Mohammad [1] proved the following generalization of Theorem A:

Theorem B: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with real positive coefficients. If $t_1\ge t_2\ge 0$ can be found such that

$$a_jt_1t_2+a_{j-1}(t_1-t_2)-a_{j-2}\ge 0,\ j=1,2,......,n+1$$
$$(a_{-1}=a_{n+1}=0)$$

then all the zeros of P(z) lie in $|z|\le t_1$.

For $t_1=1, t_2=0$, it reduces to Theorem A.

Aziz and Shah [3] proved the following more general result which includes Theorem A as a special case:

Theorem C: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n. If for some t>o,

$$\max_{|z|=R}\left|ta_0z^n+(ta_1-a_0)z^{n-1}+......+(ta_n-a_{n-1})\right|\le M$$

where R is any positive real number, then all the zeros of P(z) lie in

$$|z|\le\max\left(\frac{M}{|a_n|},\frac{1}{R}\right).$$

Recently B. A. Zargar [13] proved the following result:

Theorem D: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n. If for some real numbers $t_1,t_2;t_1\ne 0,t_1>t_2\ge 0$ and for some complex number α,

$$\max_{|z|=R}\left|\begin{matrix}\{a_n(t_1-t_2)+\alpha-a_{n-1}\}z\\ +\sum_{j=0}^{n+2}\{a_jt_1t_2+a_{j-1}(t_1-t_2)-a_{j-2}\}z^{n-j+2}\end{matrix}\right|\le M,$$

where R is a positive real number and $a_{-1}=0=a_{-2}=a_{n+1}=a_{n+2}$, then all the zeros of P(z) lie in

$$|z|\le\max\left(r,\frac{1}{R}\right),$$

where

$$r=2[|\alpha|R^2|a_n(t_1-t_2)-a_{n-1}+\alpha|+M^2]\div$$
$$[\{|\alpha|R^2M+R^2(M-|a_n|)|a_n(t_1-t_2)-a_{n-1}+\alpha|\}^2+4|a_n|R^2M$$
$$\{|\alpha|R^2|a_n(t_1-t_2)-a_{n-1}+\alpha|+M^2\}]^{\frac{1}{2}}$$
$$-[|\alpha|R^2M+R^2(M-|a_n|)|a_n(t_1-t_2)-a_{n-1}+\alpha|]$$

II. Main Results

In this paper, we prove the following generalization of Theorem D:

Theorem 1: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with $a_j=\alpha_j+i\beta_j,\quad j=0,1,2,......,n$. If for some real numbers $t_1,t_2;t_1\ne 0,t_1>t_2\ge 0$ and for some complex number α,

$$\max_{|z|=R}\left|\begin{matrix}\{\alpha_n(t_1-t_2)+\alpha-\alpha_{n-1}\}z+\\ \sum_{j=0}^{n+2}\{\alpha_jt_1t_2+\alpha_{j-1}(t_1-t_2)-\alpha_{j-2}\}z^{n-j+2}\end{matrix}\right|\le M_1,$$

$$\max_{|z|=R}\left|\begin{matrix}\{\beta_n(t_1-t_2)-\beta_{n-1}\}z+\\ \sum_{j=0}^{n+2}\{\beta_j t_1t_2+\beta_{j-1}(t_1-t_2)-\beta_{j-2}\}z^{n-j+2}\end{matrix}\right|\le M_2,$$

where R is a positive real number and $a_{-1}=0=a_{-2}=a_{n+1}=a_{n+2}$, then all the zeros of P(z) lie in

$$|z|\le\max(r_1,\frac{1}{R}),$$

where

$$\begin{aligned}r_1=2[|\alpha|R^2|a_n(t_1-t_2)-a_{n-1}+\alpha|+(M_3+M_4)^2]\div\\ [\{|\alpha|R^2(M_3+M_4)+R^2(M_3+M_4-|a_n|)\\ |a_n(t_1-t_2)-a_{n-1}+\alpha|\}^2+4|a_n|R^2(M_3+M_4)\\ \{|\alpha|R^2|a_n(t_1-t_2)-a_{n-1}+\alpha|+\\ (M_3+M_4)^2\}]^{\frac{1}{2}}-[|\alpha|R^2(M_3+M_4)\\ +R^2(M_3+M_4-|a_n|)\\ |a_n(t_1-t_2)-a_{n-1}+\alpha|]\end{aligned}$$

Remark 1: For $\beta_j=0, j=0,1,2,......,n$, Theorem 1 reduces to Theorem D.

Taking $\alpha=0$ in Theorem 1, we get the following result:

Corollary 1: Let $P(z)=\sum_{j=0}^{n}a_jz^j$ be a polynomial of degree n with $a_j=\alpha_j+i\beta_j,\quad j=0,1,2,......,n$. If for some real numbers $t_1,t_2;t_1\ne0,t_1>t_2\ge0$,

$$\max_{|z|=R}\left|\sum_{j=1}^{n+1}\{\alpha_jt_1t_2+\alpha_{j-1}(t_1-t_2)-\alpha_{j-2}\}z^{n-j+2}\right|\le M_1,$$

$$\max_{|z|=R}\left|\sum_{j=1}^{n+1}\{\beta_jt_1t_2+\beta_{j-1}(t_1-t_2)-\beta_{j-2}\}z^{n-j+2}\right|\le M_2,$$

where R is a positive real number and $a_{-1}=0=a_{-2}=a_{n+1}=a_{n+2}$, then all the zeros of P(z) lie in

$$|z|\le\max(r_1,\frac{1}{R}),$$

where

$$\begin{aligned}r_1=2(M_3+M_4)^2\div\\ [\{R^2(M_3+M_4-|a_n|)|a_n(t_1-t_2)-a_{n-1}|\}^2\\ +4|a_n|R^2(M_3+M_4)^3]^{\frac{1}{2}}\\ -[R^2(M_3+M_4-|a_n|)|a_n(t_1-t_2)-a_{n-1}|].\end{aligned}$$

III. Lemmas

For the proofs of the above results, we need the following :

Lemma 1: If f(z) is analytic for $|z|\le1$, f(0)=a, where $|a|<1$, $f'(0)=b$, $|f(z)|\le1$ for $|z|=1$, then

$$|f(z)|\le\frac{(1-|a|)|z|^2+|b||z|+|a|(1-|a|)}{|a|(1-|a|)|z|^2+|b||z|+(1-|a|)}.$$

The inequality is sharp with equality for the function

$$f(z)=\frac{a+\frac{b}{1+a}z-z^2}{1-\frac{b}{1+a}z-az^2}.$$

The above lemma is due to Govil, Rahman and Schmeisser [7].

Lemma 2: If f(z) is analytic for $|z|\le R$, f(0)=0, , $f'(0)=b$, $|f(z)|\le M$ for $|z|=R$, then

$$|f(z)|\le\frac{M|z|}{R^2}.\frac{M|z|+R^2|b|}{M+|b||z|}\text{ for }|z|\le R.$$

Lemma 2 is a simple deduction from Lemma 1.

IV. Proofs of Theorems

Proof of Theorem 1: Consider the polynomial

$$\begin{aligned}F(z)&=(t_2+z)(t_1-z)P(z)=(t_2+z)(t_1-z)(a_nz^n+\\ &+a_{n-1}z^{n-1}+......+a_1z+a_0)\\ &=-a_nz^{n+2}+\{a_n(t_1-t_2)-a_{n-1}\}z^{n+1}+\{a_nt_1t_2\\ &+a_{n-1}(t_1-t_2)-a_{n-2}\}z^n+......\\ &+\{a_2t_1t_2+a_1(t_1-t_2)-a_0\}z^2+\{a_1t_1t_2\\ &+a_0(t_1-t_2)\}z+a_0t_1t_2\end{aligned}$$

$$\begin{aligned}&=-a_nz^{n+2}+\{\alpha_n(t_1-t_2)-\alpha_{n-1}\}z^{n+1}+\{\alpha_nt_1t_2z^n\\ &+\alpha_{n-1}(t_1-t_2)-\alpha_{n-2}\}z^n+......\end{aligned}$$

$$\begin{aligned}&+......+\{\alpha_2t_1t_2+\alpha_1(t_1-t_2)-\alpha_0\}z^2+\{\alpha_1t_1t_2\\ &+\alpha_0(t_1-t_2)\}z+\alpha_0t_1t_2+i[\{\beta_n(t_1-t_2)-\beta_{n-1}\}z^{n+1}\\ &+\{\beta_nt_1t_2+\beta_{n-1}(t_1-t_2)-\beta_{n-2}\}z^n+......\\ &+\{\beta_2t_1t_2+\beta_1(t_1-t_2)-\beta_0\}z^2+\{\beta_1t_1t_2+\beta_0(t_1-t_2)\}z\\ &+\beta_0t_1t_2]\end{aligned}$$

Let $G(z) = z^{n+2}F(\frac{1}{z})$

$$= -a_n + \{\alpha_n(t_1 - t_2) - \alpha_{n-1}\}z + \{\alpha_n t_1 t_2 + \alpha_{n-1}(t_1 - t_2) - \alpha_{n-2}\}z^2 + ... + \{\alpha_2 t_1 t_2 + \alpha_1(t_1 - t_2) - \alpha_0\}z^n + \{\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2)\}z^{n+1} + \alpha_0 t_1 t_2 z^{n+2} + i[\{\beta_n(t_1 - t_2) - \beta_{n-1}\}z + \{\beta_n t_1 t_2 + \beta_{n-1}(t_1 - t_2) - \beta_{n-2}\}z^2 + ... + \{\beta_2 t_1 t_2 + \beta_1(t_1 - t_2) - \beta_0\}z^n + \{\beta_1 t_1 t_2 + \beta_0(t_1 - t_2)\}z^{n+1} + \beta_0 t_1 t_2 z^{n+2}]$$

$$= -a_n - \alpha z + H(z),$$

where

$$H(z) = \{\alpha_n(t_1 - t_2) + \alpha - \alpha_{n-1}\}z + \{\alpha_n t_1 t_2 + \alpha_{n-1}(t_1 - t_2) - \alpha_{n-2}\}z^2 + ... + \{\alpha_2 t_1 t_2 + \alpha_1(t_1 - t_2) - \alpha_0\}z^n + \{\alpha_1 t_1 t_2 + \alpha_0(t_1 - t_2)\}z^{n+1} + \alpha_0 t_1 t_2 z^{n+2} + i[\{\beta_n(t_1 - t_2) - \beta_{n-1}\}z + \{\beta_n t_1 t_2 + \beta_{n-1}(t_1 - t_2) - \beta_{n-2}\}z^2 + ... + \{\beta_2 t_1 t_2 + \beta_1(t_1 - t_2) - \beta_0\}z^n + \{\beta_1 t_1 t_2 + \beta_0(t_1 - t_2)\}z^{n+1} + \beta_0 t_1 t_2 z^{n+2}]$$

Then H(0)=0 and $|H(z)| \le M_1 + M_2$ for $|z| \le R$.

We first assume that $|a_n| \ge |\alpha|R + M_1 + M_2$.

Then , for $|z| < R$,

$$\begin{aligned} |G(z)| &= |-a_n - \alpha z + H(z)| \\ &\ge |a_n| - |\alpha||z| - (M_1 + M_2) \\ &> |a_n| - |\alpha|R - (M_1 + M_2) \\ &\ge 0 \end{aligned}$$

i.e. $|G(z)| > 0$ for $|z| < R$.

Hence, it follows that all the zeros of G(z) lie in $|z| \ge R$ in this case.

Since $F(z) = z^{n+2}G(\frac{1}{z})$, it follows that all the zeros of F(z) and therefore P(z) lie in $|z| \le \frac{1}{R}$ in case $|a_n| \ge |\alpha|R + M_1 + M_2$.

In case $|a_n| < |\alpha|R + M_1 + M_2$,we have $H(0) = 0, H'(0) = a_n(t_1 - t_2) - a_{n-1} + \alpha$ and $|H(z)| \le M_1 + M_2$ for $|z| \le R$.

Therefore, by applying Lemma 2 to H(z), we get

$$|H(z)| \le \frac{(M_1 + M_2)|z|}{R^2} \times \frac{(M_1 + M_2)|z| + |a_n(t_1 - t_2) - a_{n-1} + \alpha|R^2}{(M_1 + M_2) + |a_n(t_1 - t_2) - a_{n-1} + \alpha||z|}$$

for $|z| \le R$.

Hence, for $|z| \le R$,

$$\begin{aligned} |G(z)| &= |-a_n - \alpha z + H(z)| \\ &\ge |a_n| - |\alpha||z| - |H(z)| \\ &\ge |a_n| - |\alpha||z| - \frac{(M_1 + M_2)|z|}{R^2} \times \frac{(M_1 + M_2)|z| + |a_n(t_1 - t_2) - a_{n-1} + \alpha|R^2}{(M_1 + M_2) + |a_n(t_1 - t_2) - a_{n-1} + \alpha||z|} \\ &> 0 \end{aligned}$$

if

$$|a_n|R^2[(M_1 + M_2) + |a_n(t_1 - t_2) - a_{n-1} + \alpha||z|] - |z|[|\alpha|R^2\{(M_1 + M_2) + |a_n(t_1 - t_2) - a_{n-1} + \alpha||z|\}] - (M_1 + M_2)|z|[(M_1 + M_2)|z| + |a_n(t_1 - t_2) - a_{n-1} + \alpha|R^2] > 0$$

i.e. if

$$[|\alpha|R^2|a_n(t_1 - t_2) - a_{n-1} + \alpha| + (M_1 + M_2)^2]|z|^2 + [|\alpha|R^2(M_1 + M_2) + (M_1 + M_2)|a_n(t_1 - t_2) - a_{n-1} + \alpha|R^2 - |a_n|R^2|a_n(t_1 - t_2) - a_{n-1} + \alpha|]|z| - |a_n|R^2(M_1 + M_2) < 0$$

or if

$$[|\alpha|R^2|a_n(t_1 - t_2) - a_{n-1} + \alpha| + (M_1 + M_2)^2]|z|^2 + [|\alpha|R^2(M_1 + M_2) + R^2(M_1 + M_2 - |a_n|)|a_n(t_1 - t_2) - a_{n-1} + \alpha|]|z| - |a_n|R^2(M_1 + M_2) < 0$$

This gives $|G(z)| > 0$ if

$$|z| < \frac{1}{2[|\alpha|R^2|a_n(t_1 - t_2) - a_{n-1} + \alpha| + (M_1 + M_2)^2]} \times [\{|\alpha|R^2(M_1 + M_2) + R^2(M_1 + M_2 - |a_n|)|a_n(t_1 - t_2) - a_{n-1} + \alpha|\}^2 + 4|a_n|R^2(M_1 + M_2)\{|\alpha|R^2|a_n(t_1 - t_2) - a_{n-1} + \alpha|$$

$$+(M_1+M_2)^2\}]^{\frac{1}{2}}-[|\alpha|R^2(M_1+M_{24})$$
$$+R^2(M_1+M_2-|a_n|)|a_n(t_1-t_2)-a_{n-1}+\alpha|]$$
$$=\frac{1}{r_1}$$
$$<R$$

if

$$[\{|\alpha|R^2(M_1+M_2)+R^2(M_1+M_2-|a_n|)$$
$$|a_n(t_1-t_2)-a_{n-1}+\alpha|\}^2+4|a_n|R^2(M_1+M_2)$$
$$\{|\alpha|R^2|a_n(t_1-t_2)-a_{n-1}+\alpha|+(M_1+M_2)^2\}]^{\frac{1}{2}}$$
$$-[|\alpha|R^2(M_1+M_2)+R^2(M_1+M_2-|a_n|)$$
$$|a_n(t_1-t_2)-a_{n-1}+\alpha|]$$
$$<2R[|\alpha|R^2|a_n(t_1-t_2)-a_{n-1}+\alpha|+(M_1+M_2)^2]$$

or if

$$[\{|\alpha|R^2(M_1+M_2)+R^2(M_1+M_2-|a_n|)$$
$$|a_n(t_1-t_2)-a_{n-1}+\alpha|\}^2+4|a_n|R^2(M_1+M_2)$$
$$\{|\alpha|R^2|a_n(t_1-t_2)-a_{n-1}+\alpha|+(M_1+M_2)^2\}]$$

$$<[2R\{|\alpha|R^2|a_n(t_1-t_2)-a_{n-1}+\alpha|+(M_1+M_2)^2\}$$
$$+\{|\alpha|R^2(M_1+M_2)$$

$$+R^2(M_1+M_2-|a_n|)|a_n(t_1-t_2)-a_{n-1}+\alpha|\}]^2$$

or if

$$4|a_n|R^2(M_1+M_2)\{|\alpha|R^2|a_n(t_1-t_2)-a_{n-1}+\alpha|$$
$$+(M_1+M_2)^2\}$$

$$<4R^2\{|\alpha|R^2|a_n(t_1-t_2)-a_{n-1}+\alpha|+(M_1+M_2)^2\}^2$$
$$+4R\{|\alpha|R^2(M_1+M_2)+R^2(M_1+M_2-|a_n|)$$
$$|a_n(t_1-t_2)-a_{n-1}+\alpha|\}$$
$$\{|\alpha|R^2|a_n(t_1-t_2)-a_{n-1}+\alpha|$$
$$+(M_1+M_2)^2\}$$

or

$$|a_n|R^2(M_1+M_2)<R^2\{|\alpha|R^2|a_n(t_1-t_2)-a_{n-1}+\alpha|$$
$$+(M_1+M_2)^2\}$$
$$+R\{|\alpha|R^2(M_1+M_2)$$
$$+R^2(M_1+M_2-|a_n|)$$
$$|a_n(t_1-t_2)-a_{n-1}+\alpha|\}$$

or

$$|a_n|(M_1+M_2)<|\alpha|R^2|a_n(t_1-t_2)-a_{n-1}+\alpha|+$$
$$(M_1+M_2)^2+R\{|\alpha|(M_1+M_2)$$
$$+(M_1+M_2-|a_n|)$$
$$|a_n(t_1-t_2)-a_{n-1}+\alpha|\}$$

or

$$|a_n|(M_1+M_2)<(M_1+M_2)(|\alpha|R+M_1+M_2)$$
$$+|a_n(t_1-t_2)-a_{n-1}+\alpha|$$
$$\{|\alpha|R^2+(M_1+M_2-|a_n|)\}$$

which is true since $|a_n|<|\alpha|R+M_1+M_2$.

Thus, $|G(z)|>0$ if $|z|<\frac{1}{r_1}$. Hence, all the zeros of G(z) lie in $|z|\geq\frac{1}{r_1}$.

Since $F(z)=z^{n+2}G(\frac{1}{z})$, it follows that all the zeros of F(z) lie in $|z|\leq r_1$. But the zeros of P(z) are also the zeros of F(z). Therefore all the zeros of P(z) lie in $|z|\leq r_1$.

Combining the above two arguments, it follows that all the zeros of P(z) lie in $|z|\leq\max(r_1,\frac{1}{R})$, completing the proof of Theorem 1.

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