NEGATIVE NUMBERS & COMPLEMENT ARITHMETIC IN COMPUTER SYSTEMS

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Abstract

This paper develops a sound theory of Complement Arithmetic which is used to perform arithmetic addition operations at the Arithmetic Logic Unit of any computer. It also examines the proofs that complements of negative numbers are used to convert subtraction operations into addition operations thereby fulfilling the basic requirements of a computer to perform its functions most easily. The preceding sections to the main objective of this paper deals with the fundamental principles of Number systems that leads to what is known today as the Decimal Number System using Arabic symbols.

Key words—Arithmetic Logic Unit, **Complement Arithmetic, Radix complement.**

1. Introduction

Right from primary schools, we learn how to count 1, 2, 3, … up to 10 without any knowledge of other number systems except the Roman Numerals (I, II, III, IV, etc). The knowledge of these Roman Numerals is also limited. Another unconscious thing we do as regards numbers is the feelings that these numerals 1, 2, 3, … are natural with man. It is as if man knew them from birth; forgetting that it was a part of the early leaning. As a result of this experience, it does not become immediately obvious to us that computers must be given this same knowledge before they can differentiate these numerals and used them appropriately.

The numerals 1, 2, 3, up to 10 and beyond are known as Arabic numerals and it was probably based on the fact that human beings have ten fingers. No wonder, fingers were the first counting machine used at kindergartens even up till the present days. But there are more features in human body that point to 2-symbol-number system rather that 10-symbol-number system. Such features are: 2 hands, 2 legs, 2 eyes, 2 ears, 2 lips, 2 nostrils, etc. In fact, the entire human body is symmetrical longitudinally with each of these features present in both halves.

If number/counting has been based on other parts of human body other than the fingers, the number system that would have been most popular would have been the 2-symbol-number system referred to as a Binary Number System. This is because it is the one that perfectly fits into human life where there are only two opposing phenomena all the time; namely, good/bad, God/evil, long/short, white/black, on/off, positive/negative, etc. There are no intermediate stages by nature. People have lived to create these intermediate stages to fit into the 10-symbol number system we are stuck with over the years until the advent of computer technology.

1.1 Number

The searching question is “What is a number”? From the above introduction, one is likely to say a number is any one of the numerals 1, 2, 3, … stated above whose primary objective or purpose is to count quantities. Therefore, the generic definition of a number can therefore be given as follows:

A number is a symbol used to count quantities of things as an alphabet is used to form words of different languages.

Thus, the numerals 1, 2, 3, … are nothing but symbols. Other symbols can be use other than 1, 2, 3, …. The next follow-up question is “What is the range of quantity to be counted or what are the limits, (lower & upper) of quantities to be counted?”

The least quantity to be counted will be “nothing” or “empty” or “absence of quantity”, which is represented by a null symbol (say 0) while the upper limit could be very large – uncountable which is mathematically represented as infinity, ∞. Therefore, the range of quantity to be counted is (0 - ∞) = ∞, infinity.

Note that the null symbol and the infinity symbol used in the above are the well known conventional symbols. Other symbols can be used instead.

From the foregoing, the total quantity of symbols required to be used to count will be infinity which is impracticable because it will be difficult to remember all the symbols. Human memory capacity cannot handle such large information. Therefore, a finite quantity that can be remembered is more reasonable to adopt. Different people of the world can adopt different finite quantity of symbols to obtain a Number System to be used. The finite quantity of symbols (say r) is known as the BASE or RADIX of the Number System, (say B_r) so obtained. This, therefore, gives rise to many Number Systems.

With the finite quantity of symbols, how can these symbols be sufficient to count quantities that are greater in number than the quantity of symbols?

This is achieved by combination of the symbols in a particular fashion.

There are many ways by which combinations can be made and this is also responsible for the existence of many Number Systems.

1.2 Generic Number Systems

From the foregoing discussion, the pertinent question to answer is:

"What is a Number System"?

> A number system is a group of symbols meant to count quantities. These symbols are positioned relative to one another in a particular fashion, specifying the sequence/order of combination of these symbols to count beyond the total number of symbols of the number system.

That is, a Number System is not only an arrangement of symbols but its peculiar features must be stated to fully define a Number System. For an example, the following arrangement of symbols and its peculiar features define the Number System specified as:

α β γ δ ε …………… Fundamental symbols

Features of the Number System are as follows:

- Null Symbol = α
- Unit Symbol = β
- Highest Symbol = ε
- BASE or RADIX is (β + ε = βα). Therefore, it is a $B_{(\beta\alpha)}$ NUMBER SYSTEM. That is, the BASE of a number system is defined as:
-

> BASE of a Number System = unit symbol + highest symbol of the Number System. This is also equivalent to the quantity of the fundamental symbols making up the Number System

- ε > δ > γ > β > α
- (ε-δ) = (δ-γ) = (γ-β) = (β- α) = unit = β.
  That is, the unity symbol of this number system = β
- Combination sequence must be as in Table 1.1.

Table 1.1: Counting in $B_{(\beta+\varepsilon)}$ Number System with Greek Alphabets as Symbols

Null Symbol	Unit Symbol			Highest Symbol	REMARKS
'α'	'β'	'γ'	'δ'	'ε'	**Fundamental symbols**
'βα'	'ββ'	'βγ'	'βδ'	'βε'	Double symbol Line with 'β'
'γα'	'γβ'	'γγ'	'γδ'	'γε'	Double symbol Line with 'γ'
'δα'	'δβ'	'δγ'	'δδ'	'δε'	Double symbol Line with 'δ'
'εα'	'εβ'	'εγ'	'εδ'	'εε'	Double symbol Line with 'ε'

Note that instead of the Greek alphabets used as symbols, Arabic Numerals could be used to obtain the same Number System as follows:

0 1 2 3 4

Features of the number system are as follows:

- BASE or RADIX is (1+4 = 5). Therefore, it is an B_5, PENTAGON NUMBER SYSTEM.
- 4 > 3 > 2 > 1 > 0
- (4-3) = (3-2) = (2-1) = (1- 0) = unit = 1.
- Combination sequence must be as in Table 1.2

Table 1.2: Counting in B_5 Number System with Arabic Numerals as Symbols

Null Symbol	**Unit Symbol**			**Highest Symbol**	**REMARKS**
0	1	2	3	4	**Fundamental symbols**
10	11	12	13	14	Double symbol Line with ‘1’
20	21	22	23	24	Double symbol Line with ‘2’
30	31	32	33	34	Double symbol Line with ‘3’
40	41	42	43	44	Double symbol Line with ‘4’

It can be seen from Table-1.1 and Table-1.2 that the values of the symbols depend on their relative positions to one another. For an example, the combined symbols, δδ has its first symbol, ‘δ’ on the right hand side carrying a multiplying factor of $(\beta+\varepsilon)^\alpha = \beta$ and the second symbol, ‘δ’ from the right hand side carrying a multiplying factor of $(\beta+\varepsilon)^\beta$. This means that combined symbols in this type of Number Systems are influenced by the relative positions of their symbols, thereby making each combined symbols to be expressed as follows:
$(\delta\delta)_{(\beta+\varepsilon)} = \delta(\beta+\varepsilon)^\alpha + \delta(\beta+\varepsilon)^\beta = N_{(\beta+\varepsilon)}$ ………………….. (1.1a).
$(\delta\delta)_{\beta\alpha} = \delta(\beta\alpha)^\alpha + \delta(\beta\alpha)^\beta = N_{\beta\alpha}$ ………………….. (1.1b).
Where N is a symbol or combination of symbols, what we normally referred to as a figure.

As explained above, there are many Number Systems but for communication from one group of people to another, it is important that the same Number Systems must be used in order to promote easy understanding among different people. Universally, a Decimal Number System has been agreed upon by many nations of the world and it has become the common language among these nations without the need of interpretation. However, three other Number Systems (Octal Number System, Hexadecimal Number System and Binary Number System) have become relevant since the advent of computer technology.

2. Representation of Negative Numbers

Negative symbols/combination of symbols in any Number System are symbols less than the null symbol in the Number System under consideration. In the case of a Decimal Number System, a negative number is a number less than zero, (0). From the definition given above, such symbols do not exist since no symbol exists beyond the null symbol (the first symbol on the Number System – see Tables 1.1 & 1.2). However, a negative symbol is a mathematical concept that represents an equivalent symbol placed in an equivalent position in the opposite direction to the sequence of the symbols of the Number System. That is, the symbols of any Number System can be arranged from left to right (positive direction) and/or from right to left (negative direction) as depicted in Fig 2.1. The symbols arranged from left to right are referred to as positive numbers/symbols while those arranged from right to left are negative numbers/symbols.

The mathematical concept of negative numbers arose from insufficiency of MAN which was first experienced at the Garden of Eden as a result of sin, disobedience. Before God created MAN all what he would have needed were already created by God (Gen 2: 4 – 17). Hence, the idea of borrowing which signifies negative numbers would not have arisen if MAN had not sinned. Hence, original plan of God is what is found in the book of Deuteronomy.

> Deuteronomy 28:12; The LORD shall open unto thee his good treasure, the heaven to give the rain unto thy land in his season, and to bless all the work of thine hand: and thou shalt lend unto many nations, and thou shalt not borrow.

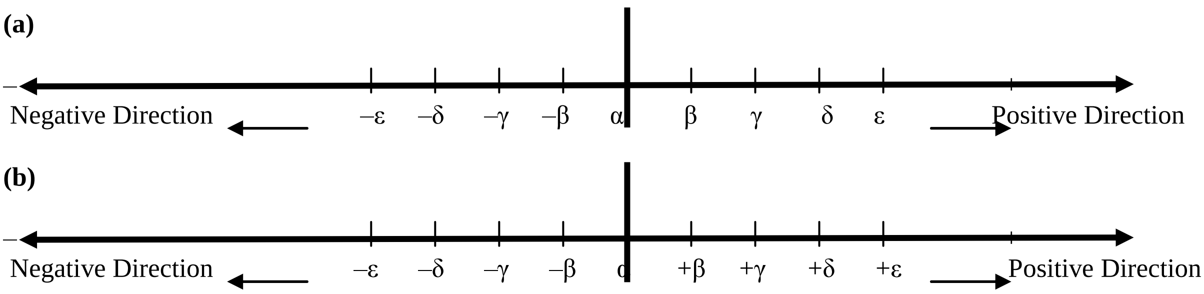


Fig 2.1: Graphical Representation of a Number System showing Positive & Negative Symbols

Note that the null symbol, 'α' is the origin where positive & negative symbols meet. The symbols to the left of the origin are combined with another symbol, (-) while the symbols to the right are not combined with any other symbol as depicted in Fig 2.1a. Therefore, symbols to the left of the origin can be said to be **signed numbers/symbols** while the ones to the right are **unsigned numbers/symbols**. The option to combine the symbols to the right of the origin with an additional symbol, (+) to signify that they are positive numbers/symbols can be employed as shown in Fig 2.1b, thus making the positive numbers/symbols signed numbers/symbols like its negative counterparts. Hence, numbers/symbols can therefore be classified as signed and unsigned numbers/symbols as shown in Fig 2.2.

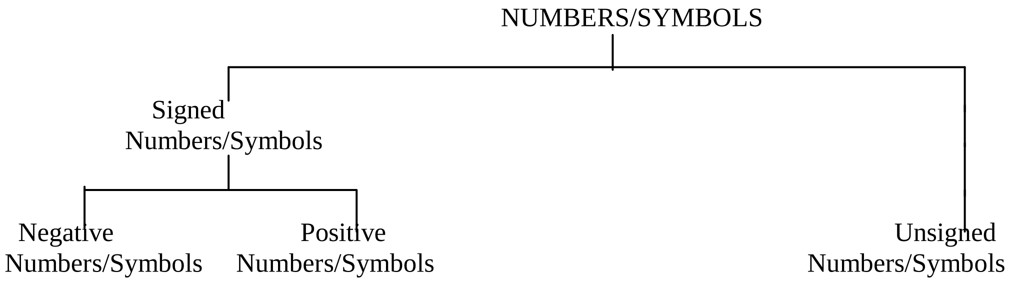


Fig 2.2.: Classification of Numbers/Symbols

There are different ways employed in computer industries to represent these negative symbols. No symbol is actually negative in any sense of reality as explained above.

If symbols/combined symbols in any Number System carry additional sign to signify a required arithmetic operation such as minus (-), multiplication (×), division (÷) and/or plus (+), it is only logical that in a Binary Number System, these signs are required to be coded in the language understood by computer machines. There are different ways of achieving this. Four of such ways are shown in Fig 2.3 but the main focus of this paper is the complement method.

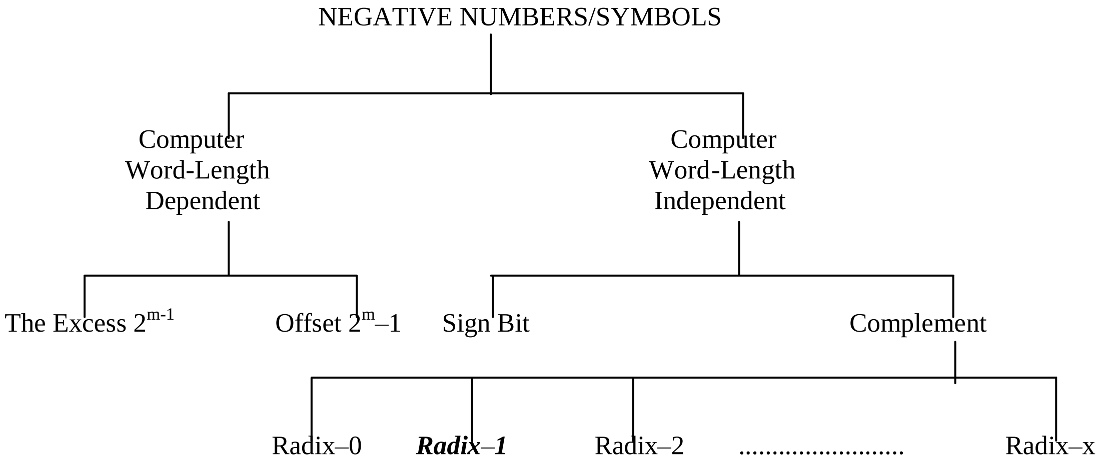


Fig 2.6: Classification of Negative Number Representation

2.1 Radix-1 Complement Method

Radix-1 complement of a number, '$\overline{N}_r$' with a base of 'r' is defined as another number, $(N_r)_{r-1}$ obtained by subtracting each digit of 'N_r' from '(r-1)'. That is,

$$\overline{N}_r = (N_r)_{r-1} \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots.. (2.1)$$

Let $N_r = (n_1 n_2 \ldots.. n_i)_r$. Where i = total number of digits that make up 'N_r'.

By definition, we have

$$(N_r)_{r-1} = (r-1-n_1)(r-1-n_2)(r-1-n_3) \dots\dots\dots (r-1-n_i) \dots\dots\dots (2.2)$$

2.2 Radix Complement Method

Radix complement of a number, '$\bar{M}_r$' with a base of 'r' is defined as another number, $(M_r)_r$ obtained by subtracting each digit of 'M_r' from '(r-1)' and adding '1'. That is,

$$\bar{M}_r = (M_r)_{r-1} + 1 \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots .. (2.3)$$

Let $M_r = (m_1 m_2 \ldots .. m_j)_r$. Where j = total number of digits that make up 'M_r'.
By definition, we have

$$(M_r)_{r-1} = (r-1-m_1)(r-1-m_2)(r-1-m_3) \ldots (r-1-m_j) + 1 \ldots \ldots \ldots (2.4)$$

Examples of Radix-1 and Radix complement of numbers in different number systems are shown in Table 2.1 and Table 2.2.

TABLE 2.1: Table of Samples of Complement of Numbers

NUMBER SYSTEMS (FIGURES)	RADIX-1 COMPLEMENT	RADIX COMPLEMENT
BINARY, B_2 **[0 1]** **$(10011)_2$**	(1-1)(1-0)(1-0)(1-1)(1-1) = 01100 **$(01100)_{2-1}$**	(1-1)(1-0)(1-0)(1-1)(1-1) +1 = 01101 **$(01101)_2$**
OCTAL, B_8 **[0, 1, 2….7]** **$(675)_8$**	(7–6)(7–7)(7–5) = 102 **$(102)_{8-1}$**	(7–6)(7–7)(7–5) + 1 = 103 **$(103)_8$**
DECIMAL, B_{10} **[0,1,2..9]** **$(154)_{10}$**	(9–1)(9–5)(9–4) = 845 **$(845)_{10-1}$**	(9–1)(9–5)(9–4) + 1 = 846 **$(846)_{10}$**
HEX, B_{16} **[0, 1,2..A,B..F]** **$(1EB)_{16}$**	(15–1)(15–14)(15–11) = E14 **$(E14)_{16-1}$**	(15–1)(15–14)(15–11) + 1 = E15 **$(E15)_{16}$**

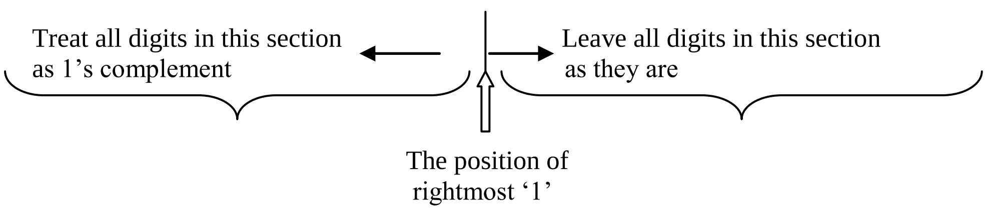
TABLE 2.2: Table of Samples of Complement of Numbers in a Number System with Symbols other than 0,1,2,3…

NUMBER SYSTEMS (FIGURES)	RADIX-1 COMPLEMENT	RADIX COMPLEMENT
$B_{(\beta+\varepsilon)}$ **[α β γ δ ε]** $(\beta\delta\gamma)_{(\beta+\varepsilon)} = (\beta\delta\gamma)_{\beta\alpha}$	Radix–1 = β+ε–β = ε's complement $(\varepsilon-\beta)(\varepsilon-\delta)(\varepsilon-\gamma) = (\delta\beta\gamma)_\varepsilon$ **$(\delta\beta\gamma)_\varepsilon$**	Radix = β+ε (δβγ)+β = δβδ **$(\delta\beta\delta)_{\beta\alpha}$**
$B_{(b+g)}$ **[a b c d e f g]** $(fegc)_{(b+g)} = (fegc)_{ba}$	Radix–1 = b+g-b = g's complement $(g-f)(g-e)(g-g)(g-c) = (bcae)_g$ **$(bcae)_g$**	Radix–1 = b+g–b = g (bcae)+b = bcad **$(bcad)_{ba}$**

Note from Table 2.1, there is a short-cut technique for obtaining Radix-1 (1's) complement and Radix (2's) complement of binary numbers as follows:

1's Complement: Change the bits to their complements.
That is, 0s are changed to 1s and 1s are changed to 0s.

2's Complement: Leave the rightmost 0s and 1as they are; then change the bits to their complements. That is, 0s are changed to 1s and 1s are changed to 0s.
OR
Identify the rightmost '1' first and from this point to the right, retain all digits (1s & 0s) as they are and change all others to the left of the rightmost 1 from 0 to 1 and from 1 to 0.



2.3 Radix-2 Complement Method

Radix-2 complement of a number, '$\bar{K}$' with a base of 'r' is defined as another number, $(K_r)_{r-2}$ obtained by subtracting each digit of 'K_r' from '(r-1)' and subtracting '1'. That is,

$$\bar{K}_r = (K_r)_{r-1} + 1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.5)$$

Let $K_r = (k_1k_2\ldots\ldots k_h)_r$. Where h = total number of digits that make up 'K_r'.
By definition, we have

$$(K_r)_{r-2} = (r-1-k_1)(r-1-k_2)(r-1-k_3)\ldots(r-1-m_h) - 1 \ldots\ldots\ldots (2.6)$$

2.4 Radix-x Complement Method

Radix-x complement of a number, '$\bar{P}$' with a base of 'r' is defined as another number, $(P_r)_{r-x}$ obtained by subtracting each digit of 'P_r' from '(r-1)' and subtracting 'x-1'. That is,

$$\bar{P}_r = (P_r)_{r-x} - (x-1) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.7)$$

Where, $x \leq r - 1$.
Let $P_r = (p_1p_2\ldots\ldots p_g)_r$. Where g = total number of digits that make up 'P_r'.
By definition, we have

$$(K_r)_{r-2} = (r-1-p_1)(r-1-p_2)(r-1-p_3)\ldots\left(r-1-p_g\right) - (x-1) \ldots\ldots\ldots (2.8)$$

In general, (R-1±y)'s Complement = (R-1)'s Complement ± y. That is,

For y = 0, (R-1)'s Complement = (R-1)'s Complement + 0 Example: 24_{10}. (R-1)'s = 9's = 75
For y = 1, (R)'s Complement = (R-1)'s Complement + 1 Example: 24_{10}. (R)'s = (R-1)'s + 1 = 75 + 1 = 76
For y = 2, (R+1)'s Complement = (R-1)'s Complement + 2 Example: 24_{10}. (R+1)'s = (R-1)'s + 2 = 75 + 2 = 77
For y = 3, (R+2)'s Complement = (R-1)'s Complement + 3 Example: 24_{10}. (R+2)'s = (R-1)'s + 3 = 75 + 3 = 78
For y = 4, (R+3)'s Complement = (R-1)'s Complement + 4 Example: 24_{10}. (R+3)'s = (R-1)'s + 4 = 75 + 4 = 79

For y = -1, (R-2)'s Complement = (R-1)'s Complement – 1 Example: 24_{10}. (R-1)'s = (R-1)'s - 1 = 75 – 1 = 74
For y = -2, (R-3)'s Complement = (R-1)'s Complement – 2 Example: 24_{10}. (R-1)'s = (R-1)'s - 2 = 75 – 2 = 73
For y = -3, (R-4)'s Complement = (R-1)'s Complement – 3 Example: 24_{10}. (R-1)'s = (R-1)'s - 3 = 75 – 3 = 72
For y = -4, (R-5)'s Complement = (R-1)'s Complement – 4 Example: 24_{10}. (R-1)'s = (R-1)'s - 4 = 75 – 4 = 71
For y = -5, (R-6)'s Complement = (R-1)'s Complement – 5 Example: 24_{10}. (R-1)'s = (R-1)'s - 5 = 75 – 5 = 70

This is diagrammatically expressed in Fig 2.7 where (R-1)'s Complement is the origin and every other complements to the left and right are a single step away as shown. Since the complement of numbers are useful only for negative numbers, only the negative values of 'x' are useful.

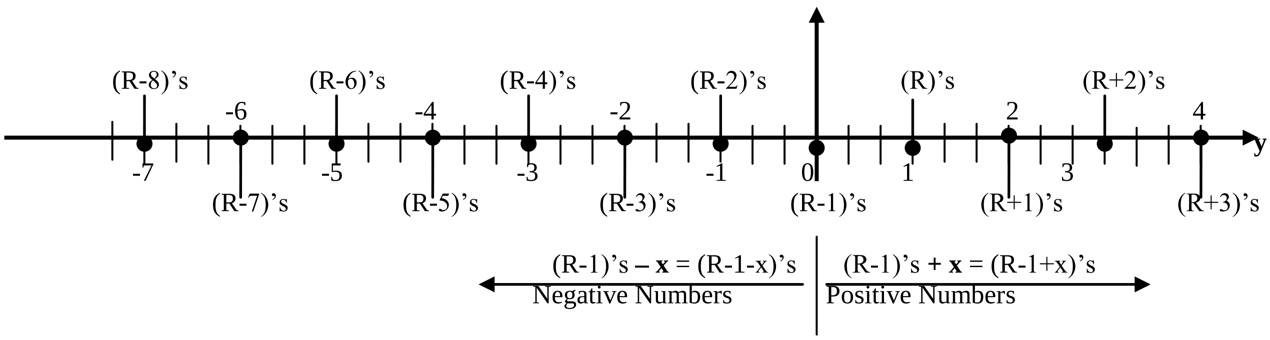


Fig 2.7: Different Complements of Negative Number Representation

In a Binary Number System (BNS), the possible complements available to the system are shown in Table 2.3.

Table 2.3: Complements in BNS

Radix-x	**Formula**	**0**	**1**
Radix-0	(R-1)+1	10	1
Radix-1	(R-1)+0	1	0
Radix-10	(R-1)- 1	0	

In a Decimal Number System (DNS), the possible complements available to the system are presented in Table 2.4.

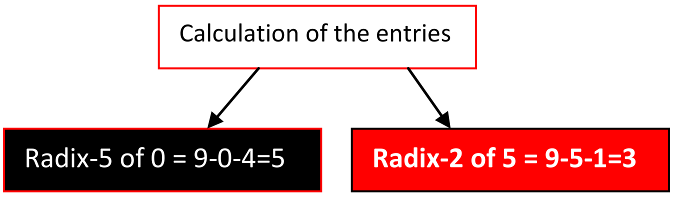


Table 2.4: Complements in DNS

Radix-x	**Formula**	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Radix-0	(R-1) +1	10	9	8	7	6	5	4	3	2	1
Radix-1	(R-1) +0	9	8	7	6	5	4	3	2	1	0
Radix-2	(R-1) – 1	8	7	6	5	4	3	2	1	0	No Complement
Radix-3	(R-1) – 2	7	6	5	4	3	2	1	0	No Complement	
Radix-4	(R-1) – 3	6	5	4	3	2	1	0	No Complement		
Radix-5	(R-1) – 4	5	4	3	2	1	0	No Complement			
Radix-6	(R-1) – 5	4	3	2	1	0	No Complement				
Radix-7	(R-1) – 6	3	2	1	0	No Complement					
Radix-8	(R-1) – 7	2	1	0	No Compleme						
Radix-9	(R-1) – 8	1	0								
Radix-10	(R-1) – 9	0									

The condition required to meet Complement Arithmetic of A – B is
A + Radix-x complement of B ≥ Radix (if A > B). That is, Radix-x Complement of B ≥ Radix – A
A + Radix-x complement of B ≤ Radix (if A < B). That is, Radix-x Complement of B ≤ Radix – A
Radix-10 is to be avoided because it is a combination of two symbols.
Note the following:

- Radix-2 complement of 9 does not exist
- Radix-3 complement of 8 & 9 do not exist
- Radix-4 complement of 7, 8 & 9 do not exist
- Radix-5 complement of 6, 7,8 & 9 do not exist
- Radix-6 complement of 5, 6, 7, 8 & 9 do not exist
- Radix-7 complement of 4, 5, 6, 7, 8, & 9 do not exist
- Radix-8 complement of 3, 4, 5, 6, 7, 8, & 9 do not exist
- Radix-9 complement of 2, 3, 4, 5, 6, 7, 8 & 9 do not exist
- Radix-10 complement of 1, 2, 3, 4, 5, 6, 7, 8 & 9 do not exist

From the above analysis, Radix-10 is the least useful while Radix-0 & Radix-1 are the most useful. Hence, Radix-0 & Radix-1 complements are the most popularly applied to represent negative numbers.

3. Complement Arithmetic

Let us assume that a negative number can be represented by its complement. That is,
(– N) = Complement of (N) which can be mathematically written as
$-N = \bar{N} \ldots \ldots \ldots \ldots (3.1)$
When Equation (3.1) is used to convert arithmetic subtraction to addition operations in such a manner that the same result is obtained, the process that is carried out is what is called Complement Arithmetic, which is a proof that validate Equation (3.1). A number of examples are presented in this section.

3.1 Example-1: Solve ($9_{10} - 7_{10}$) using Complement Arithmetic

To solve A – B by Complement Arithmetic, Equation (a) be satisfied.
(Radix-x)'s Complement of B ≥ Radix – A if A > B .... (a)

Considering (Radix-0)'s Complement:
Since A = 9 and B = 7, let us examine the required condition as given by Equation (a) where x = 0, Radix = 10 and since 9 > 7.
(Radix-0)'s Complement of 7 = 9 – 7 + 1 = 3 .... (a1)
Radix – 9 = 10 – 9 = 1 ....(a2)
The required condition: Equation (a1) ≥ Equation (a2). That is, 3 > 1.

Hence, (Radix-0)'s Complement supports Complement Arithmetic in this instance.

Considering (Radix-1)'s Complement:
Since A = 9 and B = 7, let us examine the required condition as given by Equation (a) where x = 1, Radix = 10 and since 9 > 7.
(Radix-1)'s Complement of 7 = 9 – 7 = 2 .... (a1)
Radix – 9 = 10 – 9 = 1 ....(a2)
The required condition: Equation (a1) ≥ Equation (a2). That is, 2 > 1.
Hence, (Radix-1)'s Complement supports Complement Arithmetic in this instance.

Considering (Radix-2)'s Complement:
Since A = 9 and B = 7, let us examine the required condition as given by Equation (a) where x = 2, Radix = 10 and since 9 > 7.
(Radix-2)'s Complement of 7 = 9 – 7 – 1 = 1 .... (a1)
Radix – 9 = 10 – 9 = 1 ....(a2)
The required condition: Equation (a1) ≥ Equation (a2). That is, 1 = 1.
Hence, (Radix-2)'s Complement supports Complement Arithmetic in this instance.

Considering (Radix-3)'s Complement:
Since A = 9 and B = 7, let us examine the required condition as given by Equation (a) where x = 3, Radix = 10 and since 9 > 7.
(Radix-3)'s Complement of 7 = 9 – 7 – 2 = 0 .... (a1)
Radix – 9 = 10 – 9 = 1 ....(a2)
The required condition: Equation (a1) ≥ Equation (a2). That is, 0 < 1. The required condition is violated. Hence, (Radix-3)'s Complement fails to support Complement Arithmetic in this instance.

Considering (Radix-4)'s Complement:
Since A = 9 and B = 7, let us examine the required condition as given by Equation (a) where x = 4, Radix = 10 and since 9 > 7.
(Radix-4)'s Complement of 7 = 9 – 7 – 3 = – 1 .... (a1)
Radix – 9 = 10 – 9 = 1 ....(a2)
The required condition: Equation (a1) ≥ Equation (a2). That is, – 1 < 1. The required condition is violated. Hence, (Radix-3)'s Complement fails to support Complement Arithmetic in this instance.

Alternatively, from the Table 2.4, the complements of 7 that are possible are Radix-0, Radix-1 & Radix-2. That is, the possible allowable (Radix-x)'s Complements that support Complement Arithmetic in any given situation can be read directly from Table 2.4. These three complements are employed to solve the given arithmetic subtraction operation ($\mathbf{9_{10} - 7_{10}}$).

Radix-0 Option
$9_{10} - 7_{10}$. = 9 + (Radix)'s Complement of 7 = 9 + (9 – 7 + 1) = 9 + 3 = 12
There is an overflow, meaning that the final answer is positive. Hence, answer = 2+0 = **2**

Radix-1 Option
$9_{10} - 7_{10}$. = 9 + (Radix-1)'s Complement of 7 = 9 + (9 – 7) = 9 + 2 = 11
There is an overflow, meaning that the final answer is positive. Hence, answer = 1+1 = **2**

Radix-2 Option
$9_{10} - 7_{10}$. = 9 + (Radix-2)'s Complement of 7 = 9 + (9 – 7 – 1) = 9 + 1 = 10
There is an overflow, meaning that the final answer is positive. Hence, answer = 0+2 = **2**

This principle can be extended to any number system. Table 3.1 shows the proof of such practice.

Table 3.1a
Complements in Base Three (B_3) System

Radix-x	**Formula**	**0**	**1**	**2**
Radix-0	(R-1) +1	10	2	1
Radix-1	(R-1) +0	2	1	0
Radix-2	(R-1) – 1	1	0	
Radix-10	(R-1) – 2	0		

Table 3.1b
Complements in Base Four (B_4) System

Radix-x	**Formula**	**0**	**1**	**2**	**3**
Radix-0	(R-1) +1	10	3	2	1
Radix-1	(R-1) +0	3	2	1	0
Radix-2	(R-1) – 1	2	1	0	
Radix-3	(R-1) – 2	1	0		
Radix-10	(R-1) – 3	0			

Table 3.1c
Complements in Base Five (B_5) System

Radix-x	**Formula**	**0**	**1**	**2**	**3**	**4**
Radix-0	(R-1) +1	10	4	3	2	1
Radix-1	(R-1) +0	4	3	2	1	0
Radix-2	(R-1) – 1	3	2	1	0	
Radix-3	(R-1) – 2	2	1	0		
Radix-4	(R-1) – 3	1	0			
Radix-10	(R-1) – 4	0				

Table 3.1d: Complement Arithmetic in Different Number Systems

$5_{10} - 3_{10}$		$101_2 - 011_2$		$12_3 - 10_3$		$11_4 - 03_4$		$10_5 - 03_5$	
Radix-1 = (9)	Radix-0 = (10)	Radix-1 = (1)	Radix-0 = (2)	Radix-1 = (2)	Radix-0 = (3)	Radix-1 = (3)	Radix-0 = (4)	Radix-1 = (4)	Radix-0 = (5)
9's Comp of 3 = 6	10's Comp of 3 = 7	1's Comp of 011 = 100	2's Comp of 011 = 101	2's Comp of 10 = 12	2's Comp of 10 = 13	3's Comp of 03 = 30	4's Comp of 03 = 31	4's Comp of 03 = 41	5's Comp of 03 = 42
+5 +6 11 +1 2_{10}	+5 +7 12 +0 2_{10}	+101 +100 1001 +1 010_2	+101 +101 1010 +0 010_2	+12 +12 101 +1 02_3	+12 +13 102 +0 02_3	+11 +30 101 +1 02_3	+11 +31 102 +0 02_3	+10 +41 101 +1 02_3	+10 +42 102 +0 02_3
$5_{10} - 3_{10} = 101_2 - 011_2 = 12_3 - 10_3 = 11_4 - 03_4 = 10_5 - 03_5$									

Table 3.1e: Complement Arithmetic in Different Number Systems

$5_{10} - 3_{10}$		$101_2 - 011_2$		$12_3 - 10_3$		$11_4 - 03_4$		$10_5 - 03_5$	
Radix-2 = (8)	Radix-3 = (7)	Radix-2 X	Radix-3 X	Radix-2 = (1)	Radix-3 X	Radix-2 = (2)	Radix-3 = (1)	Radix-2 = (3)	Radix-3 = (2)
5	4			01	13	2?	1?	30	2?
+5 +5 10 +2 2_{10}	+5 +4 9 ?????	These Complements do not exist (see Table 2.3).		+12 +01 20 ?????		? These Complements of 3_4 do not exist (see Table 3.1b).		+10 +30 40 ?????	? (Radix-3)'s Complement of 3 does not exist (see Table 3.1c).

????? Rules of Complement Arithmetic are violated. Hence, (Radix-2)'s and (Radix-3)'s Complements are not feasible. That is,
(Radix-3)'s Complement of $3_{10} \geq$ Radix – 5_{10} ($4_{10} \geq 10_{10} - 5_{10} = 5_{10}$). This is violated.
(Radix-2)'s Complement of $10_3 \geq$ Radix – 12_3 ($01_3 \geq 10_3 - 12_3 = ?$). This is violated.
(Radix-3)'s Complement of $03_5 \geq$ Radix – 10_5 ($30_5 \geq 10_5 - 10_5 = 0_5$). This is violated.

This proves the universality of (Radix-0)'s & (Radix-1)'s Complements.
X Radix Complement so marked does not exist (see appropriate Tables 2.3, 3.1a,b,c).

3.2 Example-2: Solve ($7_{10} - 9_{10}$) using Complement Arithmetic

To solve A – B by Complement Arithmetic, Equation (b) must be satisfied.
(Radix-x)'s Complement of B ≤ Radix – A if A < B .... (b)

Considering (Radix-0)'s Complement:

Since A = 7 and B = 9, let us examine the required condition as given by Equation (b) where x = 0, Radix = 10 and since 7 < 9.

(Radix-0)'s Complement of 9 = 9 – 9 + 1 = 1 .... (b1)
Radix – 7 = 10 – 7 = 3 ....(b2)
The required condition: Equation (b1) ≤ Equation (b2). That is, 1 < 3.
Hence, (Radix-0)'s Complement supports Complement Arithmetic in this instance.

Considering (Radix-1)'s Complement:
Since A = 7 and B = 9, let us examine the required condition as given by Equation (b) where x = 1, Radix = 10 and since 7 < 9.
(Radix-1)'s Complement of 9 = 9 – 9 = 0 .... (b1)
Radix – 9 = 10 – 7 = 3 ....(b2)
The required condition: Equation (b1) ≤ Equation (b2). That is, 0 < 3.
Hence, (Radix-1)'s Complement supports Complement Arithmetic in this instance.

Considering (Radix-2)'s Complement:
Since A = 7 and B = 9, let us examine the required condition as given by Equation (b) where x = 2, Radix = 10 nd since 7 > 9.
(Radix-2)'s Complement of 9 = 9 – 9 – 1 = – 1 .... (b1)
Radix – 9 = 10 – 9 = 1 ....(b2)
The required condition: Equation (b1) ≤ Equation (b2). That is, – 1 < 1.
Hence, (Radix-2)'s Complement fails to support Complement Arithmetic in this instance because Equation (b1) is negative. That is, (Radix=2)'s Complement does not exist as indicated in Table 2.4.

From Table 2.4, the feasible (Radix-x)'s Complement that will bring about solution to this arithmetic subtraction operation is only (Radix-0)'s Complement while (Radix-1)'s Complement is found feasible from the calculation. It is therefore safer to use the entries in Table 2.4. These two complements are employed to solve the given arithmetic subtraction operation (**7_{10} – 9_{10}**).

Radix-0 Option
$7_{10} – 9_{10}$. = 7 + (Radix)'s Complement of 9 = 7 + (9 – 9 + 1) = 7 + 1 = 8
There is no overflow, meaning that the final answer is negative. Hence, let the answer = – y
Using Equation 3.1, we have

$$-y = \bar{y} \ldots \ldots \ldots \ldots (3.1)$$

Therefore,

$$\bar{y} = 8 \ldots \ldots \ldots \ldots (3.2)$$

Complement both sides of Equation (3.2), we have

$$y = (Radix - 0)'s\ Complement\ of\ 7 = 9 - 8 + 1 = 2 \ldots \ldots \ldots \ldots (3.3)$$

Hence, the answer = **– 2**

Radix-1 Option
$7_{10} – 9_{10}$. = 7 + (Radix-1)'s Complement of 9 = 7 + (9 – 9) = 7 + 0 = 7
There is no overflow, meaning that the final answer is negative. Hence, let the answer = – y
Using Equation 3.1, we have

$$-y = \bar{y} \ldots \ldots \ldots \ldots (3.1)$$

Therefore,

$$\bar{y} = 7 \ldots \ldots \ldots \ldots (3.2)$$

Complement both sides of Equation (3.2), we have

$$y = (Radix - 1)'s\ Complement\ of\ 7 = 9 - 7 = 2 \ldots \ldots \ldots \ldots (3.3)$$

Hence, the answer = **– 2**

Radix-2 Option
$7_{10} – 9_{10}$. = 7 + (Radix-2)'s Complement of 9 = 7 + (9 – 9 – 1) = – 1
Since (Radix-2)'s Complement is negative, no Complement Arithmetic is possible.

Similarly, this principle can be extended to any number system. Table 3.2 shows the proof of such practice using Table 3.1a, b & c.

Table 3.2d: Complement Arithmetic in Different Number Systems									
$7_{10} - 9_{10}$		$0111_2 - 1001_2$		$021_3 - 100_3$		$13_4 - 21_4$		$12_5 - 14_5$	
Radix-1 = (9)	Radix-0 = (10)	Radix-1 = (1)	Radix-0 = (2)	Radix-1 = (2)	Radix-0 = (3)	Radix-1 = (3)	Radix-0 = (4)	Radix-1 = (4)	Radix-0 = (5)
9's Comp of 9 = 0	10's Comp of 9 = 1	1's Comp of 1001 = 0110	2's Comp of 1001 = 0111	2's Comp of 100 = 122	2's Comp of 100 = 200	3's Comp of 21 = 12	4's Comp of 21 = 13	4's Comp of 14 = 30	5's Comp of 14 = 31
+7 +0 7 $\bar{y} = 7$ -2_{10}	+7 +1 8 $\bar{y} = 8$ -2_{10}	+0111 +0110 1101 $\bar{y} = 1101$ $= 0010_2$	+0111 +0111 1110 $\bar{y} = 1110$ $= 0010_2$	+021 +122 220 $\bar{y} = 220$ $= 002_3$	+021 +200 221 $\bar{y} = 221$ $= 002_3$	+13 +12 31 $\bar{y} = 31$ $= 02_3$	+13 +13 32 $\bar{y} = 32$ $= 02_3$	+12 +30 42 $\bar{y} = 42$ $= 02_3$	+12 +31 43 $\bar{y} = 43$ $= 02_3$
$7_{10} - 9_{10} = 0111_2 - 1001_2 = 021_3 - 100_3 = 13_4 - 21_4 = 12_5 - 14_5$									

Alternatively, Complement Arithmetic can be graphically implemented. This is demonstrated to solve (i) $5_{10} - 3_{10}$ and (ii) $3_{10} - 5_{10}$ as follows:

Solution: (i) Solve (5_{10} - 3_{10}) using 9's complement.
Draw two parallel line (L_1 and L_2). L_1 is labelled with positive numbers while L_2 bear both positive and negative numbers as shown in Fig 3.1.
Step-1: Mark the positive number, 5 on L_1
Step-2: Mark the negative number, –3 on L_2
Step-3: Transfer the negative number perpendicularly to L_1. The corresponding value is the 9's Complement of the negative number (9's Complement of 3 = 6).
Step-4: Add the two numbers (5+6 = 11) so marked on L_1. Transfer the answer, 11 perpendicularly to L_2 and mark the value as the final answer.

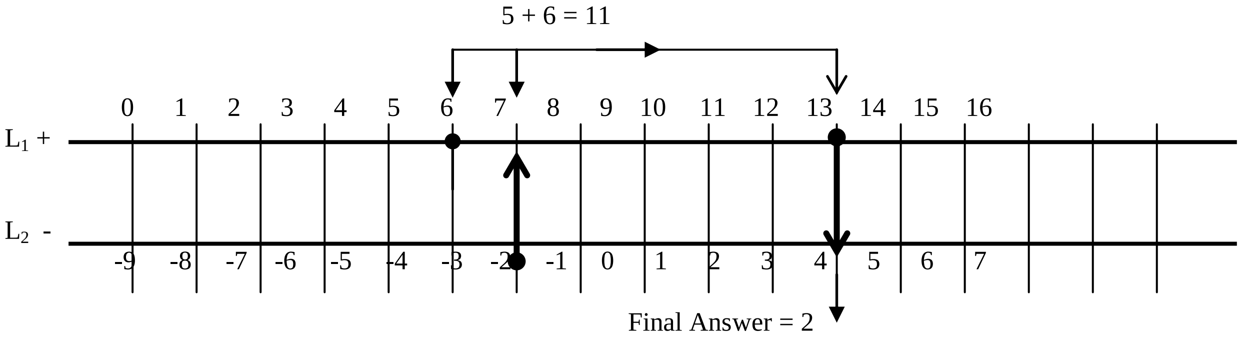


Fig 3.1: Graphical 9's Complement Arithmetic - Radix-1 Complement Arithmetic

Solution: (i) Solve (3_{10} - $5_{10)}$ using 9's Complement.
Draw two parallel line (L_1 and L_2). L_1 is labelled with positive numbers while L_2 bear both positive and negative numbers as shown in Fig 3.2.
Step-1: Mark the positive number, 3 on L_1
Step-2: Mark the negative number, –5 on L_2
Step-3: Transfer the negative number perpendicularly to L_1. The corresponding value is the 9's complement of the negative number (9's complement of 5 = 4).
Step-4: Add the two numbers (3+4 = 7) so marked on L_1. Transfer the answer, 7 perpendicularly to L_2 and mark the value as the final answer.

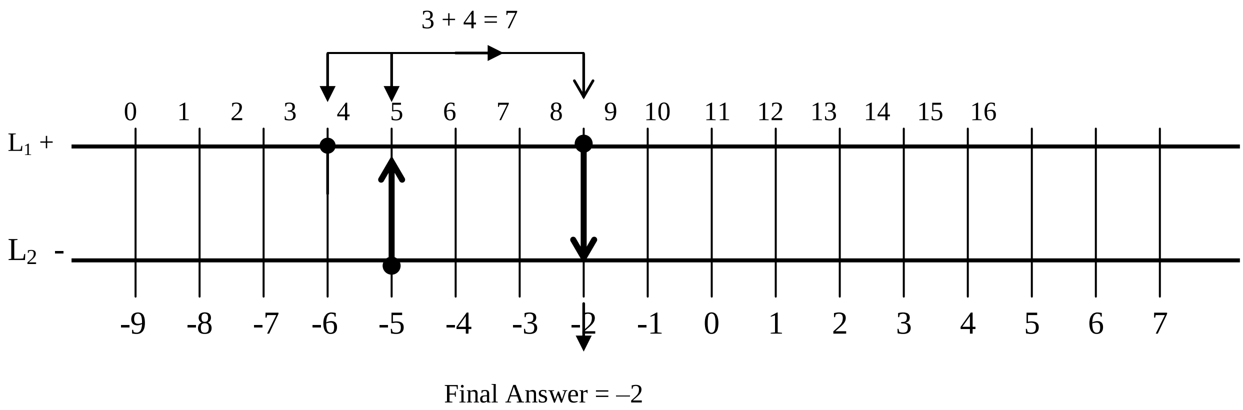


Fig 3.2: Graphical 9's Complement Arithmetic – Radix-1 Complement Arithmetic

Solution: (i) Solve ($5_{10} – 3_{10}$) using 10's Complement.
Draw two parallel line (L_1 and L_2). L_1 is labelled with positive numbers while L_2 bear both positive and negative numbers as shown in Fig 3.3.
Step-1: Mark the positive number, 5 on L_1
Step-2: Mark the negative number, –3 on L_2
Step-3: Transfer the negative number perpendicularly to L_1. The corresponding value is the 9's complement of the negative number (10's complement of 3 = 7).
Step-4: Add the two numbers (5+7 = 12) so marked on L_1. Transfer the answer, 12 perpendicularly to L_2 and mark the value as the final answer.

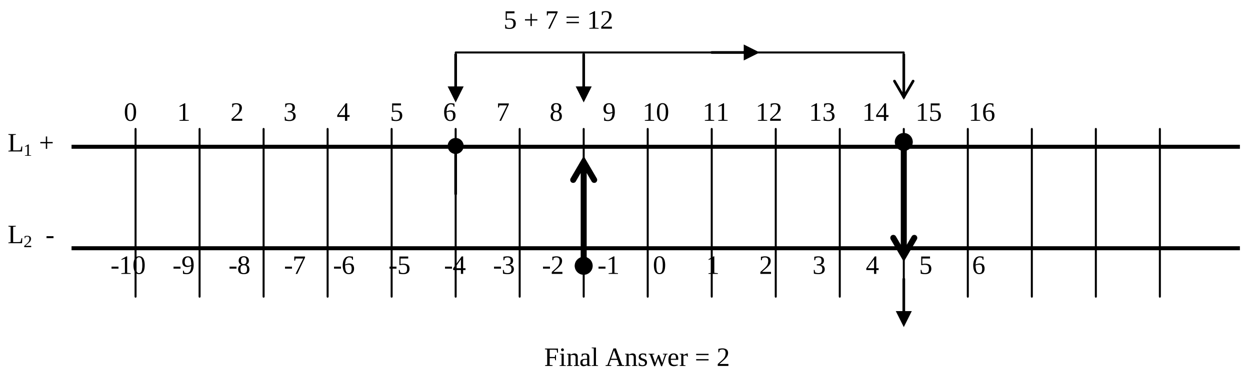


Fig 3.3: Graphical 10's Complement Arithmetic - Radix Complement Arithmetic

Solution: (i) Solve ($3_{10} – 5_{10}$) using 10's Complement.
Draw two parallel line (L_1 and L_2). L_1 is labelled with positive numbers while L_2 bear both positive and negative numbers as shown in Fig 3.4.
Step-1: Mark the positive number, 3 on L_1
Step-2: Mark the negative number, –5 on L_2
Step-3: Transfer the negative number perpendicularly to L_1. The corresponding value is the 9's complement of the negative number (10's complement of 5 = 5).
Step-4: Add the two numbers (3+5 = 8) so marked on L_1. Transfer the answer, 8 perpendicularly to L_2 and mark the value as the final answer.

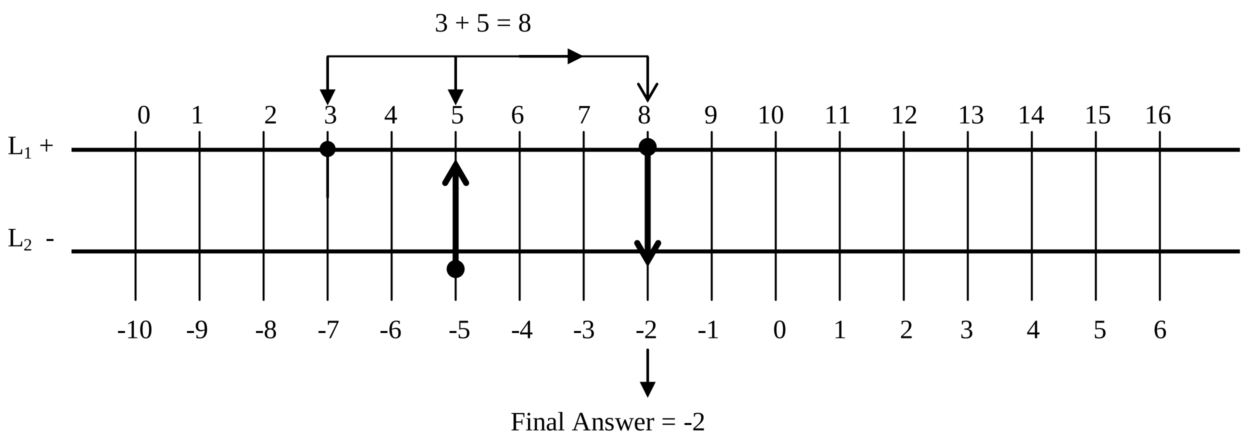


Fig 3.4: Graphical 10's Complement Arithmetic – Radix Complement Arithmetic

4. Conclusion

This paper has provided a sound theory of Complement Arithmetic and proved that complement of negative numbers can adequately be employed to perfectly represent the negative numbers in solving arithmetic subtraction operations. This is a necessary operation required at the Arithmetic Logic Unit of any computer system which is principally designed to perform addition operations only. The theory is particularly important because of wrong concept that is found in the presentations of some authors and the inability of previous concept to solve the same problems using Complement Arithmetic with (Radix-x)'s Complements where $x > 1$.

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