A Comparative Study of the Total Domination with Paired Domination

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Abstract **– In this paper an attempt is made to compare the most important parameters of the Total Domination and Paired Domination of simple undirected graphs. In the first section all the needed concepts and the earlier results are given. In the subsequent section, the proofs of the properties needed to prove the required inequalities are given. The only assumption made is that all graphs are free from isolated vertices.**

Keywords **– Total domination, claws, carona, upper total domination, claw-free graphs, paired-dominations.**

Mathematics Subject Classification **- 05c**
Field **– Graph Theory** ***Subfield*** **– Domination**

I. Introduction

A. Total Domination

A.1. General

As a preliminary, in this we give the basic concepts of Total Domination and Paired Dominations and their comparisons in the succeeding sections:

All graphs $G=(V(G),E(G))$ will be simple, finite, and undirected. Based on the context, the types of the graphs to be used will be discussed, the vertex set and edge set of G are respectively denoted by V and E.

A.2. Neighbourhoods

Let $v \in V$, be a vertex, the *neighbourhood* of v denoted by $N(v)$ is defined as $N(v)=\{w \in V : vw \in E\}$. The *closed neighbourhood* of v denoted by $N[V]$ is defined as $N[v]=N(v) \cup \{v\}$. Similarly the neighbourhood of a subset of V say $S \subseteq V$ denoted by $N(S)$ is defined as $N(S)=\{w \in V / w \in N(v), v \in S\}$ and the closed neighbourhood of S is : $N[S]=N(S) \cup S$.

A.3. Degree of a vertex

For any vertex $v \in V$, the degree of v, denoted by $d(v)$, is the number of vertices adjacent to v, or simply $d(v)=|N(v)|$. The smallest and the largest values of $d(v)$ for all $v \in V$, denoted by $\delta(G)$ and $\Delta(G)$ are respectively called the minimum degree and maximum degree of G.

The *circumference* of a graph G, is the length of a largest cycle in G and denoted by $c(G)$. A *Hamiltonian path* (*cycle*) in a graph G is a path (cycle) that passes through every vertex in V. A graph is called *Hamiltonian* if it contains a Hamilton cycle. ie if $c(G)=|V(G)|$.

A *connected graph* G is a graph for which there is a path from u to v (a $u-v$ path) for every pair of vertices $u,v \in V$. In general, G is called k*-connected* if $G-S$ is connected for any $S \subseteq V$ where $|S|<k$. The largest value of k for which G is k-connected is called the *connectivity* of G, and is denoted by $k(G)$. G is 2-connected if it is Hamiltonian. For a connected graph G, when $G-S$ is not connected for $S \subset V$, the set is called a *vertex cut* of G. If $\{v\}$ is a vertex cut of G, then v is called as a *cut vertex* of G. For any vertex set $S \subset V$, the number of components in $G-S$ is denoted by $\omega(G-S)$.

The sub graph of G induced by the non empty vertex subset $S \subseteq V$ is denoted by $G[S]$, or simply by $[S]$.

A subset $D \subseteq V$ is called a *dominating set* of a graph G if for every $v \in V$, either $v \in D$ or v is adjacent to a vertex in D, that is $N[D]=V$. The minimum cardinality of a dominating set in G is the *domination number* of G and is denoted by

$\gamma(G)$. An *independent set* in G is a set of pairwise nonadjacent vertices, and the *independence number* of G, denoted as $\beta(G)$ is the maximum cardinality of an independent set in G. A dominating set which is also an independent set is called an *independent dominating set*. The minimum cardinality of such a set in G is the *independent domination number* of G, denoted as $i(G)$. A subset D of V is called a *total dominating set* of a graph G if every vertex of G is adjacent to a vertex in D, ie, if $V = N(D)$. The minimum cardinality of a total dominating set in G is the *total domination number* of G denoted as $\gamma_t(G)$. From the definition $\gamma_t(G)$ can be defined only when $\delta(G) > 0$. Since every independent dominating set is also a dominating set

$$\gamma_t(G) - 2 \le \gamma_t(G+e) \le \gamma_t(G).$$

II. Paired Domination

We have already the minimal size of a total dominating set, the total domination number as γ_t. Now the maximal size of an inclusive minimal total dominating set , the upper total domination number, is denoted by Γ_t. A paired dominating set is a dominating set whose induced sub graph has a perfect matching. The minimal size of a paired dominating set, the paired domination number, is denoted by γ_p. The maximal size of an inclusion wise minimal paired dominating set, the upper paired domination number, is denoted by Γ_p.

III. New Definitions

Let G be a graph. If a graph H is an induced sub graph of G, then we write $H \subseteq G$. If H does not contained in G, then G is said to be H - free. If $\mathcal{H}$ is a set of graphs, then G is said to be $\mathcal{H}$- free if G is H - free for every $H \in \mathcal{H}$. A *corona* of G is a graph obtained from G by attaching a pendant vertex to each vertex of G. The corona of G is denoted as $Cr(G)$. The complete bipartite graph $K_{1,3}$ is called the *claw*. In general, graphs of the form $K_{1,r}$ are called as *generalized claws*. The path on 3 vertices we denote by P_3. We observe that $P_3 \cong K_{1,2}$. For each $r \ge 3$, the graph T_r is obtained from $K_{1,r}$ by subdividing each edge exactly once*. The claw, T_3 and the corona of the claw are depicted in Fig. 1.

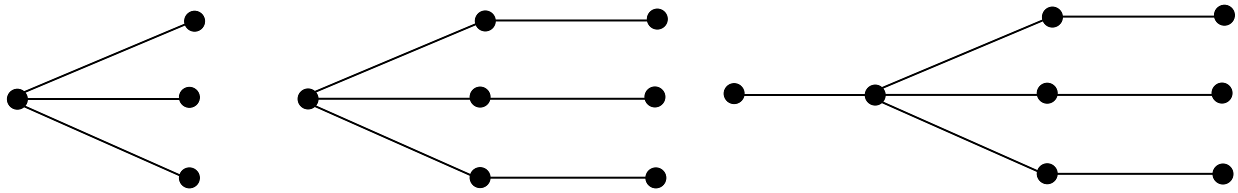

Fig. 1 $K_{1,3}$, T_3 and $Cr(K_{1,3})$

A *dominating set* X of G is a vertex subset such that any vertex of $V(G) \setminus X$ has a neighbour in X. By the definition a *total dominating set* is a dominating set X whose induced sub graph, denoted by $G[X]$, does not have isolated vertices. Since the graph we considered do not have isolated vertices, any graph has a total dominating set. An induced paired dominating set is a paired dominating set that induces a 1-regular sub graph. That is to say that it is an induced matching whose matched vertices dominate the graph. Not every graph has an induced paired dominating set (This is clear in Fig. 2 given later).

It is interesting to know how the parameters, in the two types of dominations considered, behave when both are compared.

* The isomorphic graphs are treated as identical since we deal only with graph invariants. The graphs here are finite, simple and undirected.

Usually the absolute difference of two parallel parameters is not bounded. That is the differences viz: $\gamma_p - \gamma_t$, $\Gamma_p - \Gamma_t$ and $\gamma_p - \Gamma_t$ can grow arbitrarily a seen from $Cr(K_{1,r})$. Hence the ratios of the parameters viz: γ_p / γ_t, Γ_p / Γ_t and γ_p / Γ_t are considered.

Proposition 1: Haynes and Slater [4] have shown that

$$\gamma_p(G) \le 2\gamma_t(G) - 2 \text{ and hence}$$

$$\frac{\gamma_p}{\gamma_t} \le 2 \text{ in general.}$$

Similarly since

$\gamma_p(T_r)/\gamma_t(T_r) = 2 - 2/(r+1)$ for each $r \ge 1$

(1) is asymptotically sharp.

The same result is true for Γ_p / Γ_t as in Dorbec etial [7].

Now (1) can be improved if certain sub graphs are forbidden.

According to Brigham and Dutton [6], for claw-free graphs the relation

$\gamma_p / \gamma_t < 4/3$ is true.

We now generalize this and for claw-free graphs and find the bound for γ_p / Γ_t.

IV. RESULTS NEEDED FOR ESTABLISHING NEW BOUNDS

We need some auxillary results (AR) to establish the bounds. These are known results.

AR1:

Let G be a graph. Any induced sub graph H of G has an induced paired dominating set iff G does not contain C_5, $Cr(K_3)$ or $Cr(P_3)$ as induced sub graph [8].

Here sufficient conditions for a graph to have a total dominating set that induces a generalized claw-free graph. Again if G is $Cr(K_{1,r})$- free for some $r \ge 3$, then G has a total set T such that $G[T]$ is $K_{1,r}$ - free.

AR2:

Let G be a graph. A matching M of G is maximum iff there is no augmenting path with respect to M. [9]

AR3:

If G is a k- connected $K_{1,r}$ - free graph, then

$$\text{ঌ}(G) \ge \min\left\{ \frac{k}{r+k+1}|V(G)|, \left\lfloor \frac{1}{2}|V(G)| \right\rfloor \right\}$$

as in [10] (1)

A particular result from AR3 is

R1: If G is a $K_{1,r}$ - free graph for some $r \ge 3$, then

$$\frac{\text{ঌ}(G)}{|V(G)|} \ge \frac{1}{r}$$

Proof : Let G be a $K_{1,r}$ - free graph for some $r \ge 3$.

Let G be connected. Using AR3

$$\text{ঌ}(G) \ge \min\left\{ \frac{1}{r}|V(G)|, \left\lfloor \frac{1}{2}|V(G)| \right\rfloor \right\}$$

Since $r \ge 3$ and $|V(G)| \ge 2$ (Since G does not have isolated vertices)

$$\frac{1}{r}|V(G)| \le \left\lfloor \frac{1}{2}|V(G)| \right\rfloor.$$

Hence ঌ$(G) \ge \frac{1}{r}|V(G)|$,

and this proves the result. With the above result, we prove the following:

R2: Let G be a graph with a total dominating set T such that $G[T]$ is $K_{1,r}$ - free for some $r \ge 3$. Then there is a paired dominating set P with $\frac{|P|}{|T|} \le 2 - \frac{2}{r}$.

Proof: Let G be a graph that has a total dominating set T such that $G[T]$ is a $K_{1,r}$ - free graph. We assume that T is minimal. Let M be a matching of size ঌ$(G[T])$ of $G[T]$. Since $G[T]$ is $K_{1,r}$ - free, by R1 we have

$$\frac{|M|}{|T|} \ge \frac{1}{r} \qquad \text{(A)}$$

Let $U \subseteq T$ be the set of unmatched vertices and let $u \in U$ be arbitrary. Then $G[T \setminus \{u\}]$ does not have isolated vertices, since otherwise u would necessarily be matched to one of these isolated vertices. Since T is a minimal total dominating set of G, there is a vertex set $u' \in V | T$ whose only neighbor in T is u. Otherwise $T \setminus \{u\}$ would be a total dominating set of G, too. We call u' a private neighbour of u. Let P be the set obtained from T by adding exactly one private neighbour u' for each $u \in U$. In $G[P]$, each $u \in U$ can then be matched to its former private neighbour u'

. Hence, $G[P]$ has a perfect matching and is hence a paired dominating set of G. We observe that M leaves exactly $|T|-2|M|$ vertices of T unmatched.
That is, $|U|=|T|-2|M|$. Hence,

$|P|=|T|+|U|=2|T|-2|M|$. This together with (A) gives

$$\frac{|P|}{|T|}=\frac{2|T|-2|M|}{|T|}=2-\frac{2|M|}{|T|}\le 2-\frac{2}{r}.$$

R3: Let G be a graph and let $r\ge 3$ such that any minimal total dominating set of G induces a $K_{1,r}$ - free graph. Then

$$\frac{\Gamma_p(G)}{\Gamma_t(G)}\le 2-\frac{2}{r}.$$

Proof: Let G be a graph and let $r\ge 3$ such that any minimal total dominating set of G induces a $K_{1,r}$ - free graph. Let P be a minimal paired dominating set of G and M be a perfect matching of $G[P]$. Since any paired dominating set is a total dominating set, too, there is a minimal total dominating set $T\subseteq P$. Let $M'=M\cap E(G[T])$ be the restriction of M to $G[T]$ and let $U\subseteq T$ be the vertices of T that are not matched by M'.

Let us assume that M' is not a maximum matching of $G[T]$. By the result in AR2 there is an augmenting path in $G[T]$ with respect to M'. Hence there is a bigger matching of $G[T]$, say M'', such that the set of unmatched vertices of M'', denoted by U', is a subset of U. Let P' be the set obtained from T by adding the matching partner in M of each $u\in U'$. Clearly P' is a proper subset of P. P' is a dominating set, since $T\subseteq P'$. Apart from this $G[P']$ has a perfect matching and hence a paired dominating set. This contradicts the minimality of P.

Hence M' is a maximum matching of $G[T]$. As any minimal total dominating set of G induces a $K_{1,r}$ - free graph, $G[T]$ is also $K_{1,r}$ - free. By the result in R1, $|M'|\ge T/r$. Since $|P|=|T|+|U|$ and $|U|=|T|-2|M'|$, we have on substitution

$|P|\le\left(2-\frac{2}{r}\right)|T|$. Since P is arbitrary, it proves the statement.

As a consequence of R2 and R3 , we have the Proposition 2.

Proposition 2: Let G be a $K_{1,r}$ - free graph for some $r\ge 3$. Then

$$\frac{\gamma_p(G)}{\gamma_t(G)}\le 2-\frac{2}{r} \qquad (2)$$

And this bound is sharp for each $r\ge 3$. Further ,

$$\frac{\Gamma_p(G)}{\Gamma_t(G)}\le 2-\frac{2}{r} \qquad (3)$$

based on this, we have the following results.

R4: If G is a graph with maximum degree Δ,

$$\frac{\gamma_p(G)}{\gamma_t(G)}\le 2-\frac{2}{1+\Delta} \qquad (4)$$

and this bound is sharp for each Δ. Moreover ,

$$\frac{\Gamma_p(G)}{\Gamma_t(G)}\le 2-\frac{2}{1+\Delta} \qquad (5)$$

Here (3) and (5) are not sharp possibly.

Proof : Let G be a $K_{1,r}$ - free graph for some $r\ge 3$.

By R3, (3) is true.

We have now to prove (2). For every total dominating set T , $G[T]$ is also $K_{1,r}$ - free. Hence if T is a total dominating set of size $\gamma_t(G)$, R2 gives a paired dominating set P of size at most $(2-2/r)\,\gamma_t(G)$. Thus

$$\frac{\gamma_p(G)}{\gamma_t(G)}\le\frac{|P|}{\gamma_t(G)}\le 2-\frac{2}{r}$$

To see that (2) is sharp for each r , we see that T_{r-1} is $K_{1,r}$ - free , $\gamma_p(T_{r-1})=2(r-1)$ and $\gamma_t(T_{r-1})=r$. Dividing and after simplification, we have

$$\frac{\gamma_p(T_{r-1})}{\gamma_t(T_{r-1})}=2-\frac{2}{r} \qquad (5\text{ a})$$

since the proof of (4) is direct from the result in Proposition 2, we proceed to its sharpness. When , $\Delta=1$, the case is obvious. For $\Delta\ge 2$, observe that T_Δ attains the bound. Finally (6) completes the proof.

Proposition 3:

Let G be a $\{C_5, T_r\}$- free graph for some $r\ge 3$. Then

$$\frac{\gamma_p(G)}{\gamma_t(G)} \leq 2 - \frac{2}{r} \qquad (6)$$

and

$$\frac{\Gamma_p(G)}{\Gamma_t(G)} \leq 2 - \frac{2}{r} \quad , \qquad (7)$$

both the bounds are sharp for each $r \geq 3$.

For the proof of this we need the following result in R5.

R5: Let G be a $\{C_5, T_r\}$- free graph for some $r \geq 3$. Then the sub graph induced by any minimal total dominating set of G is $K_{1,r}$ - free.

Proof : Let G be a $\{C_5, T_r\}$- free graph for some $r \geq 3$ and let T be a minimal total dominating set of G. If we assume that $G[T]$ is not $K_{1,r}$- free. That is, there is a subset $S \subseteq T$ with $G[S] \cong K_{1,r}$. Let x be the dominating vertex of $K_{1,r}$ and let $s_1, s_2, \ldots, s_r$ be the pendant vertices of the $K_{1,r}$. Since T is a minimal total dominating set, each vertex $s_i \in S \setminus \{x\}$ has a neighbour $v_i \notin S$ such that $N(v_i) \cap S = \{s_i\}$. If there are some $1 \leq i < j \leq r$ such that v_i is adjacent to v_j, then $G[x, s_i, v_i, s_j, v_j] \cong C_5$, a contradiction to the assumption on G. Hence, $\{v_i, 1 \leq i \leq r\}$ is a stable set. Thus, $G[S \cup \{v_i : 1 \leq i \leq r\}] \cong T_r$, a contradiction to the assumption on G.

Proof of proposition 3 : Let G be a $\{C_5, T_r\}$-free graph for some $r \geq 3$. By R4, (7) is true.

Now we prove (6). R5 shows that for every minimal total dominating set T, $G[T]$ is $K_{1,r}$-free. The rest is analogous to the proof of Proposition 2.

Sharpness is attained by $Cr(K_{1,r-1})$: we observe that

$$\gamma_p(Cr(K_{1,r-1})) = \Gamma_p(Cr(K_{1,r-1})) = 2r-2$$

where as

$$\gamma_t(Cr(K_{1,r-1})) = \Gamma_t(Cr(K_{1,r-1})) = r.$$

Proposition 4:

Let G be a graph. The following statements are equivalent:

(i)Any induced sub graph H of G has an induced paired dominating set.

(ii) max $\gamma_p(H) \,|\, \Gamma_t(H) = 1$

(iii) G is $\{C_5, Cr(K_3), Cr(P_3)\}$- free

The forbidden sub graphs of proposition 4 are displayed below in Fig 2.

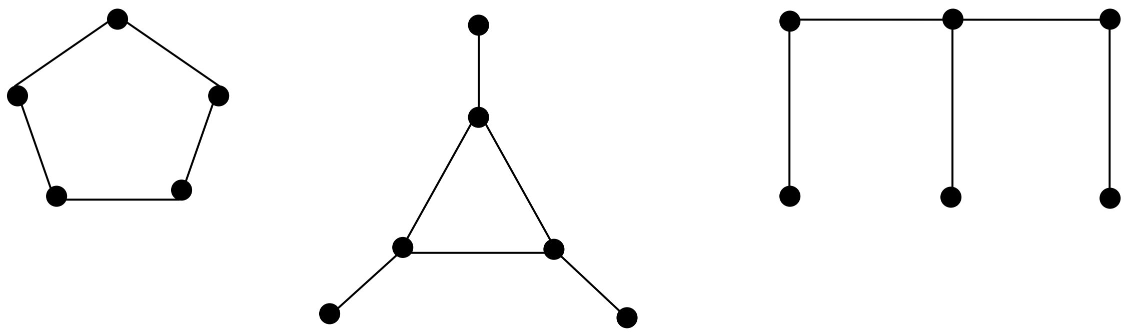

Fig. 2 C_5, $Cr(K_3)$ and $Cr(P_3)$

To prove the proposition 4 we need the following :

R6: For any G,

$$\frac{\gamma_p(Cr(G))}{\Gamma_t(Cr(G))} = 2 - 2\,\frac{ঌ(G)}{|V(G)|} \qquad (8)$$

In particular, for any r,

$$\frac{\gamma_p(Cr(K_{1,r}))}{\Gamma_t(Cr(K_{1,r}))} = 2 - \frac{2}{r+1} \qquad (9)$$

Proof of R6: Let G be a graph. In $[1]$ it is shown that

$$\gamma_p(Cr(G)) = 2\,|V(G)| - 2\,ঌ(G).$$

$\Gamma_t\left(Cr(G)\right) = \left|V(G)\right|$. These two facts leads to (8). The validity of (9) follows from (8), $\gamma_p(K_{1,r}) = 1$ and $\left|V(K_{1,r})\right| = r+1$.

Apart from this , we have

$$\frac{\gamma_p(C_5)}{\Gamma_t(C_5)} = \frac{\gamma_p\left(Cr\left(K_3\right)\right)}{\Gamma_t\left(Cr\left(K_3\right)\right)} = \frac{\gamma_p\left(Cr\left(P_3\right)\right)}{\Gamma_t\left(Cr\left(P_3\right)\right)} = \frac{4}{3} \quad (10)$$

Proof of Proposition 4: By AR1, conditions (ii) and (iii) are equivalent.

Now let G be a graph, to see that condition 1 implies condition 2, assume that any induced sub graph H of G has an induced paired dominating set. Hence, H has a paired dominating set which is a minimal total dominating set. Thus $\gamma_p(H) \le \Gamma_t(H)$. By (10), any graph that contains C_5, $Cr\left(K_3\right)$, $Cr\left(P_3\right)$ as induced sub graph does not meet condition (ii). Hence condition (ii) implies condition (iii) and this completes the proof.

Proposition 5 deals with the ratio $\gamma_p \left| \Gamma_t \right.$. It completely determines the maximal value of the ratio $\gamma_p \left| \Gamma_t \right.$ taken over the induced sub graphs of a graph. Also , it provides a complete list of possible values and gives a finite forbidden sub graph characterization for each value.

Proposition 5: Let G be a graph and let

$$\lambda = \max\left\lfloor 2, \min\left\{r : G \ is\ Cr\left(K_{1,r}\right) - free\right\}\right\rfloor \quad (11)$$

then

$$\max_{H\subseteq G} \frac{\gamma_p(H)}{\Gamma_t(H)} = \begin{cases} 2-\frac{2}{\lambda}, & if\ G\,is\ \left\{C_5, Cr\left(K_3\right)\right\} - \\ \max\left\{\frac{4}{3}, 2-\frac{2}{\lambda}\right\}, & otherwise \end{cases} \quad (12)$$

The possible values of $\max_{H\subseteq G} \gamma_p(H) \left| \Gamma_t(H) \right.$ this proposition 5 are given in Table 1.

TABLE 1

THE VALUES OF $\max_{H\subseteq G} \frac{\gamma_p(H)}{\Gamma_t(H)}$ PROVIDED BY PROPOSITION 5

Property of Graph G	$\max_{H\subseteq G} \gamma_p(H) \left\| \Gamma_t(H) \right.$
$\left\{C_5, Cr\left(K_3\right), Cr\left(P_3\right)\right\} -$ free	1
$Cr\left(K_{1,3}\right) -$ free , not $\left\{C_5, Cr\left(K_3\right), Cr\left(P_3\right)\right\} -$ free	4/3
$Cr\left(K_{1,4}\right) -$ free , but $Cr\left(K_{1,3}\right) \subseteq G$	3/2
$Cr\left(K_{1,5}\right) -$ free , but $Cr\left(K_{1,4}\right) \subseteq G$	8/5
$Cr\left(K_{1,6}\right) -$ free , but $Cr\left(K_{1,5}\right) \subseteq G$	5/3
$Cr\left(K_{1,7}\right) -$ free , but $Cr\left(K_{1,6}\right) \subseteq G$	12/7
	..
$Cr\left(K_{1,r}\right) -$ free , but $Cr\left(K_{1,r-1}\right) \subseteq G$	2-2/r

As a consequence of this proposition 5, we have the following bounds:

R7: Let G be a $Cr\left(K_{1,r}\right) -$ free graph for some $r \ge 3$, then

$$\frac{\gamma_p(G)}{\Gamma_t(G)} \le 2-\frac{2}{r} \quad (13)$$

This bound is sharp for each $r \ge 3$

In particular , we obtain

R8: Let G be connected graph with maximum degree $\Delta \ge 2$ that is not isomorphic to C_5. Then

$$\frac{\gamma_p(G)}{\Gamma_t(G)} \le 2-\frac{2}{\Delta} \quad (14)$$

This bound is sharp for each $\Delta \ge 2$.

Proof of proposition 5: (Combining R2 and Proposition 5 and 9 , we get Proposition 5)

Let G be a graph and λ be as defined in (11)

First assume that $\lambda = 2$ ie G is $Cr\left(P_3\right) -$ free

Again assume that G is $\{C_5, Cr(K_3)\}-$ free. Then by Proposition 4 , proved,

$\gamma_p(H) \leq \Gamma_t(H)$ holds for any induced sub graph H of G. Hence, (12) holds in this case.

We again assume that G is not $\{C_5, Cr(K_3)\}-$free

By (10) ,

$$\max_{H \subseteq G} \frac{\gamma_p(H)}{\Gamma_t(H)} \geq \frac{4}{3} \quad (15)$$

Since $\lambda = 2$, G is $Cr(K_{1,2})-$ free by definition of λ. Now let H be any induced sub graph of G. In particular, $Hi\ Cr(K_{1,3})-$ free. Hence by proposition 9, H has a minimal total dominating set T such that $G[T]$ is $K_{1,3}-$ free. By R2, H has a paired dominating set with $|P|/|T| \leq 4/3$.

Hence ,

$$\frac{\gamma_p(H)}{\Gamma_t(H)} \leq \frac{|P|}{|T|} \leq \frac{4}{3} \quad (16)$$

Combining (15) and (16) , we get

$$\max_{H \subseteq G} \frac{\gamma_p(H)}{\Gamma_t(H)} = \frac{4}{3}$$; which is the desired equality (12) for the case when $\lambda = 2$.

Now assume that $\lambda \geq 3$. Then, G is not $Cr(K_{1,2})$-free and hence (12) gives

$$\max_{H \subseteq G} \frac{\gamma_p(H)}{\Gamma_t(H)} \geq \frac{4}{3}$$

For completing the proof, we have prove that

$$\max_{H \subseteq G} \frac{\gamma_p(H)}{\Gamma_t(H)} = 2 - \frac{2}{\lambda} \quad (17)$$

By definition of λ , G is $Cr(K_{1,\lambda})$ - free. Let H be any induced sub graph of G. Then H is also $Cr(K_{1,r})$ - free . Hence by Proposition 9, H has a minimal total dominating set T such that $G[T]$ is $K_{1,\lambda}$ - free. By R2, H has a paired dominating set P with $\frac{|P|}{|T|} \leq 2 - \frac{2}{\lambda}$. Hence, $\gamma(H)/\Gamma_t(H) \leq 2 - \frac{2}{\lambda}$ and hence

$$\max_{H \subseteq G} \frac{\gamma_p(H)}{\Gamma_t(H)} = 2 - \frac{2}{\lambda} \quad (18)$$

On the other hand, G contains $Cr(K_{1,\lambda-1})$ as induced sub graph. R5 gives

$$\gamma_p(Cr(K_{1,\lambda-1})) / \Gamma_t(Cr(K_{1,\lambda-1})) = 2 - \frac{2}{\lambda}.$$

Hence

$$\max_{H \subseteq G} \frac{\gamma_p(H)}{\Gamma_t(H)} \geq 2 - \frac{2}{\lambda} \quad (19)$$

Now (18) and (19) give (17). This completes the proof.

Proof of R7 : Proposition 5 gives (13) . sharpness is obtained by $Cr(K_{1,r-1})$, as R6 shows.

Proof of R8 : Let G be a connected graph of maximum degree $\Delta \geq 2$ that is isomorphic to C_5. If $\Delta = 2$, G is a path of length at least 2 or a cycle that is not C_5 and hence (14) holds by Proposition 4. Sharpness is obtained by P_3 , since $\gamma_p(P_3) = 2$. If $\Delta \geq 3$, G is $Cr(K_{1,\Delta})$ - free. R7 then provides (14). Sharpness is obtained by $Cr(K_{1,\Delta-1})$, as R6 shows.

V. CONCLUSION

In this work a comparison is made on the most important parameters of the Total Domination and Paired Domination of simple undirected graphs. All the needed concepts and the earlier results are discussed. The proofs of the properties needed to prove the required inequalities are describbed.

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