FREE VIBRATION ANALYSIS OF LAMINATED COMPOSITE PLATES BY FINITE ELEMENT METHOD

Mr. Sharda Prasada[#1], Mr. Mahesh Dewangan[*2]

[#] ***M.E Student, Associate Professor***

#Deptt. Of Mechanical Engineering,

Shri Shankaracharya Group of Institution, Bhilai, Chhattisgarh – India

[1]sharda_prasada@yahoo.com

[2]*Mahesh_k_d@yahoo.com*

Abstract - Laminated composites have found extensive use as aircraft structural materials due to their high strength to weight and stiffness to weight ratios. In addition to aircraft structures, they have found their way into many automobile and building structures. Apart from having better strength, stiffness and lower weight properties, they have better corrosion resistance, thermal and acoustic insulation properties than metallic structures. In the present investigation, the free vibration analysis of 3-ply symmetric cross-ply laminated square plate subjected to different sets of boundary condition is carried out using Finite Element Method. The fundamental natural frequencies of free vibration are obtained for various side to thickness ratio of the laminated composite plate. A computer program in MATLAB has been developed for the vibration analysis on the basis of finite element formulation. Also, a commercially available finite-element package ANSYS is used for the numerical analysis.

***Keywords*— Laminated composites plates, free vibration, cross-ply, FEM, ANSYS, CPT, FSDT**

I. INTRODUCTION.

Composite materials can be classified into two groups such as filled materials and reinforced materials. The main feature of filled materials is the existence of some basic or matrix material whose properties are improved by filling it with some particles. Usually the matrix volume fraction is more than 50% in such materials. The basic components of reinforced materials (sometimes referred to as advanced composites) are long and thin fibers possessing high strength and stiffness. The fibers are bound with a matrix material whose volume fraction is less than 50%.

Fibers used in advanced composites are of two types: (i) carbon fibers (e.g. carbon, boron, steel, glass, aramid, polyethylene fibers etc.) and (ii) natural fibers (e.g. wood, coir, bamboo, wool, cotton, rice, natural silk, asbestos etc.). Fig.1.1 shows different types of fabrications in composites.

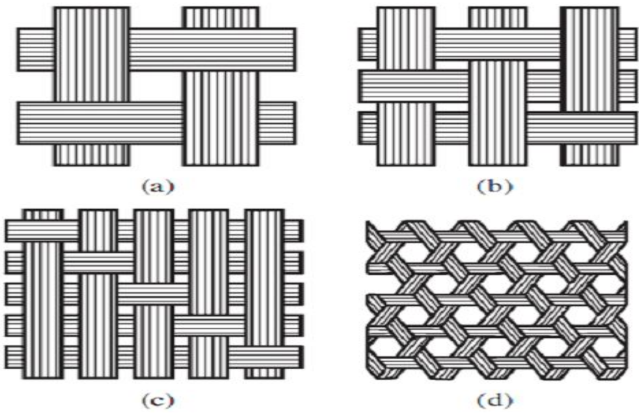

Fig. 1.1 : Different types of fabrications in composites (a) Plain , (b)twill, (c) biaxial woven, and (d) triaxial woven fabrics

The increasing demand for light weight yet strong and stiff structures has led to the development of advanced fiber-reinforced composites. These materials are used not only in the aerospace industry but also in a variety of commercial applications in the automobile, marine and biomedical areas. Traditionally, fibrous composites are manufactured by laminating several layers of unidirectional fiber tapes pre-impregnated with matrix material. The effective properties of the composite can be controlled by changing several parameters like the fiber orientation in a layer, stacking sequence, fiber and matrix material properties and fiber volume fraction. However, the manufacture of fibrous laminated composites from prepregs is labor intensive. Laminated composites also lack through-thickness reinforcement, and hence have poor inter laminar strength and fracture toughness.

2. THEORY OF COMPOSITES.

The analysis of plate and shell structures are mainly based on the following theories:

1. The classical plate theory (CPT)
2. The first-order shear deformation theory (FSDT)
3. The higher-order shear deformation theory (HSDT)

2.1 Elastic constitutive equations for different materials

Composites are a subclass of anisotropic materials that are classified as orthotropic. Orthotropic materials have properties that are different in three mutually perpendicular directions. They have three mutually perpendicular axes of symmetry, and a load applied parallel to these axes produces only normal strains. However, loads that are not applied parallel to these axes produce both normal and shear strains. Therefore, orthotropic mechanical properties are a function of orientation. Fig. 2.1 demonstrates the directions of stress and strain components in three mutually perpendicular directions.

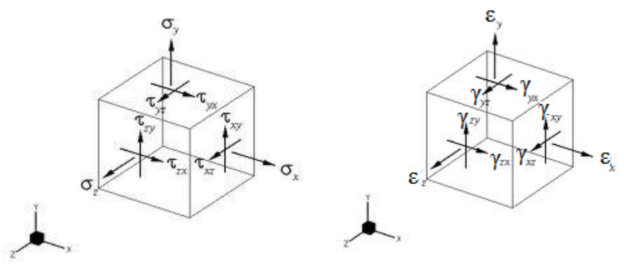

Fig. 2.1: stress and strain components in 3-Dimensions

2.1.1 Stress-strain relations (3-Dimensional)

The 3-Dimensional stress-strain relations for different materials are given by Generalized Hooke's law. In matrix form,

$$\{\sigma\}=[C]\{\varepsilon\} \qquad \text{.........(2.1.1)}$$

where [C] is called stiffness matrix.

general anisotropic material (no plane of material symmetry)

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{xz}\\ \tau_{xy}\end{Bmatrix}=\begin{bmatrix}C_{11}&C_{12}&C_{13}&C_{14}&C_{15}&C_{16}\\ C_{21}&C_{22}&C_{23}&C_{24}&C_{25}&C_{26}\\ C_{31}&C_{32}&C_{33}&C_{34}&C_{35}&C_{36}\\ C_{41}&C_{42}&C_{43}&C_{44}&C_{45}&C_{46}\\ C_{51}&C_{52}&C_{53}&C_{54}&C_{55}&C_{56}\\ C_{61}&C_{62}&C_{63}&C_{64}&C_{65}&C_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{yz}\\ \gamma_{xz}\\ \gamma_{xy}\end{Bmatrix} \quad \text{.... (2.1.2)}$$

In general, there are (6x6=36) number of unknowns in the above equation. But due to symmetry, the numbers of unknowns reduce to 21.

For specially orthotropic materials (3-mutually perpendicular planes of material symmetry), the number of unknowns in the constitutive equation reduces to 9. Hence, the stress-strain relationship takes the form

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{xz}\\ \tau_{xy}\end{Bmatrix}=\begin{bmatrix}C_{11}&&&&&\\ C_{21}&C_{22}&&\text{Sym}&&\\ C_{31}&C_{32}&C_{33}&&&\\ 0&0&0&C_{44}&&\\ 0&0&0&0&C_{55}&\\ 0&0&0&0&0&C_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{yz}\\ \gamma_{xz}\\ \gamma_{xy}\end{Bmatrix} \quad \text{........ (2.1.3)}$$

For transversely isotropic material (an orthotropic material is called transversely isotropic when one of its principal plane is a plane of isotropy. At every point on this plane, the mechanical properties are the same in all directions), there are 5 unknown coefficients in the equation. The stress-strain relationship for such materials reduces to

$$\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{xz}\\ \tau_{xy}\end{Bmatrix}=\begin{bmatrix}C_{11}&&&&&\\ C_{21}&C_{22}&&\text{Sym}&&\\ C_{12}&C_{23}&C_{33}&&&\\ 0&0&0&\frac{C_{22}-C_{33}}{2}&&\\ 0&0&0&0&C_{55}&\\ 0&0&0&0&0&C_{55}\end{bmatrix}\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{yz}\\ \gamma_{xz}\\ \gamma_{xy}\end{Bmatrix} \quad \text{........ (2.1.4)}$$

For isotropic material (a material having infinite number of planes of material symmetry through a point), there are 2 unknown coefficients in the equation and hence, the stress and strain components are related as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{yz} \\ \tau_{xz} \\ \tau_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & & & & & \\ C_{21} & C_{11} & & & Sym & \\ C_{12} & C_{12} & C_{11} & & & \\ 0 & 0 & 0 & C_{44} & & \\ 0 & 0 & 0 & 0 & C_{44} & \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{Bmatrix} \quad \text{........ (2.1.5)}$$

where $C_{44} = \frac{C_{11} - C_{12}}{2}$

2.1.2 Strain-stress relations (3-Dimensional)

The 3-Dimensional strain-stress relations for different materials is written in matrix form as

$\{\varepsilon\}=[S]\{\sigma\}$ .... (2.1.6)

where [S] is called compliance matrix.

The stiffness matrix [C] and compliance matrix [S] are related as

$[C] = [S]^{-1}$ .... (2.1.7)

For isotropic materials, the strain-stress constitutive equation is given as

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \varepsilon_4 = \gamma_{yz} \\ \varepsilon_5 = \gamma_{zx} \\ \varepsilon_6 = \gamma_{xy} \end{bmatrix} = \begin{bmatrix} 1/E & -\nu/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & 1/E & -\nu/E & 0 & 0 & 0 \\ -\nu/E & -\nu/E & 1/E & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \sigma_4 = \tau_{yz} \\ \sigma_5 = \tau_{zx} \\ \sigma_6 = \tau_{xy} \end{bmatrix}$$

.... (2.1.8)

where the elastic modulus (E), shear modulus (G) and Poisson's ratio (ν) are related as

$G = E/2(1+\nu)$ .... (2.1.9)

For orthotropic materials, the strain-stress relation becomes

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \varepsilon_4 = \gamma_{yz} \\ \varepsilon_5 = \gamma_{zx} \\ \varepsilon_6 = \gamma_{xy} \end{bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & -\nu_{31}/E_3 & 0 & 0 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & -\nu_{32}/E_3 & 0 & 0 & 0 \\ -\nu_{13}/E_1 & -\nu_{23}/E_2 & 1/E_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/G_{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_{31} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/G_{12} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \sigma_4 = \tau_{yz} \\ \sigma_5 = \tau_{zx} \\ \sigma_6 = \tau_{xy} \end{bmatrix}$$

..... (2.1.10)

where elastic modulus $E_1 = \sigma_x / \varepsilon_x$ and so on and Poisson's ratio $\nu_{12} = -\varepsilon_y / \varepsilon_x$ and so on. Also

$\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2}$, $\frac{\nu_{13}}{E_1} = \frac{\nu_{31}}{E_3}$ and $\frac{\nu_{23}}{E_2} = \frac{\nu_{31}}{E_3}$

For transversely isotropic material (in plane yz)

$E_2 = E_3$, $G_{12} = G_{13}$

$\nu_{12} = \nu_{13}$, $G_{23} = \frac{E_2}{2(1+\nu_{23})}$

2.1.3 Stress-strain relations for arbitrary orientation of a lamina

A lamina inclined to the reference axes is shown in fig. 2.2.

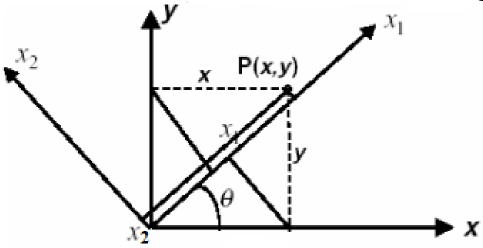

Fig. 2.2: Arbitrary orientation of a lamina

The stress-strain relation for a lamina inclined at an angle θ to the reference axes is written as

$\{\sigma\}_{xy} = [Q]_{xy}\{\varepsilon\}_{xy}$ .... (2.1.17)

where $[Q]_{xy} = [T_\sigma][Q]_{1\text{-}2}[T_\sigma]^T$ is the transformed stiffness matrix.

$$[T_\sigma]=\begin{bmatrix} m^2 & n^2 & -2mn \\ n^2 & m^2 & 2mn \\ -mn & -mn & m^2 - n^2 \end{bmatrix} \quad \text{........ (2.1.18)}$$

is called stress transformation matrix.

writing the elements of stiffness matrices $[Q]_{xy}$ and $[Q]_{1\text{-}2}$,

$$\begin{bmatrix} Q_{xx} & Q_{xy} & Q_{xs} \\ Q_{yx} & Q_{yy} & Q_{yx} \\ Q_{sx} & Q_{sy} & Q_{ss} \end{bmatrix} = [T_\sigma] \begin{bmatrix} Q_{11} & Q_{21} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} [T_\sigma]^T \quad \text{..... (2.1.19)}$$

2.1.4 Strain-stress relations for arbitrary orientation of a lamina

The strain-stress relation for a lamina inclined at an angle θ to the reference axes is written as

$\{\varepsilon\}_{xy} = [S]_{xy} \{\sigma\}_{xy}$ .... (3.2.20)

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{yx}}{E_y} & \frac{\eta_{sx}}{G_{xy}} \\ -\frac{\nu_{xy}}{E_x} & \frac{1}{E_y} & \frac{\eta_{sy}}{G_{xy}} \\ \frac{\eta_{xs}}{E_x} & \frac{\eta_{ys}}{E_y} & \frac{1}{G_{xy}} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} \quad \text{........ (3.2.21)}$$

3.1 Stress-strain relationship in laminates

A small element of laminate surrounding a point (x, y) on the geometric mid-plane at a distance z from middle surface and subjected to plane-stress conditions. The directions of stresses are also indicated.

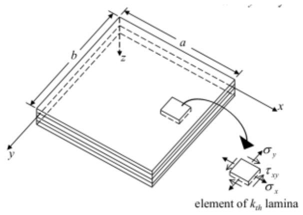

Fig 3.1: Lamina in Plane Stress (σ_x, σ_y, τ_{xy})

The strain-displacement equations for transverse shear strains and the in-plane strains at any point in the lamina in terms of strains and curvature at middle surface are given by

$$\left.\begin{aligned} \varepsilon_x &= \frac{\partial u}{\partial x} = \varepsilon_x^0 + z\kappa_x \\ \varepsilon_y &= \frac{\partial v}{\partial y} = \varepsilon_y^0 + z\kappa_y \\ \gamma_{xy} &= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = \gamma_{xy}^o + z\kappa_{xy} \end{aligned}\right\} \quad \ldots\ldots (3.1.1)$$

A general laminate consists of an arbitrary number of layers (N). The Cartesian stress components within any one of these layers, say the k[th] layer, are defined by equation

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}_k = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

$$= \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{Bmatrix} \varepsilon_x^o + z\kappa_x \\ \varepsilon_y^o + z\kappa_y \\ \gamma_{xy}^o + z\kappa_{xy} \end{Bmatrix} \quad \ldots\ldots (3.1.2)$$

Figure 3.2. The location of each layer is important in defining the governing relations for laminate response.

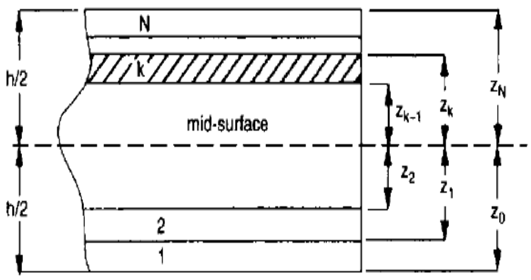

Fig. 3.2: Laminate stacking sequence nomenclatures

The stress resultants are given by

$$\left.\begin{aligned} N_x &= \int_{-h/2}^{h/2} \sigma_x \, dz \\ N_y &= \int_{-h/2}^{h/2} \sigma_y \, dz \\ N_{xy} &= \int_{-h/2}^{h/2} \tau_{xy} \, d \end{aligned}\right\} \quad \ldots\ldots (3.1.3)$$

Putting equation (3.1.3) in matrix form:

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} dz \quad \ldots\ldots (3.1.4)$$

The units of the moment resultants are moment per unit length. The moment resultants are given by

$$\left.\begin{aligned} M_x &= \int_{-h/2}^{h/2} \sigma_x \, z \, dz \\ M_y &= \int_{-h/2}^{h/2} \sigma_y \, z \, dz \\ M_{xy} &= \int_{-h/2}^{h/2} \tau_{xy} \, z \, dz \end{aligned}\right\} \quad \ldots\ldots (3.1.5)$$

Putting equation (3.1.5) in matrix from:

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} z \, dz \quad \ldots\ldots (3.3.6)$$

The integrals in equations (3.1.4) and (3.1.6) must be performed over each ply and then summed. Using the schematics of laminate in fig.3.4, equations (3.1.4) and (3.1.6) may be written as:

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k dz \quad \text{........ (3.1.7)}$$

and

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \sum_{k=1}^{n} \int_{z_{k-1}}^{z_k} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}_k zdz \quad \text{........ (3.1.8)}$$

The directions for all the stress and moment resultants are shown in the fig.3.3. The doubled- headed arrow indicates torque in a direction determined by the right-hand-rule.

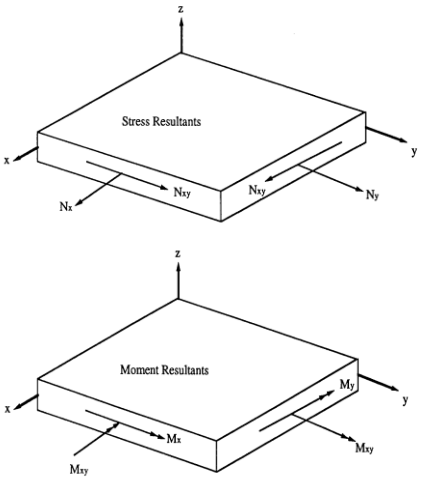

Fig. 3.3: Directions of stress and moment resultants

Substitution of equation (3.1.2) into equations (3.1.7) and (3.1.8) gives

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \sum_{k=1}^{n} \left\{ \int_{z_{k-1}}^{z_k} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} dz + \int_{h_{k-1}}^{h_k} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} k_x \\ k_y \\ k_{xy} \end{bmatrix} zdz \right\}$$

.... (3.1.9)

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \sum_{k=1}^{n} \left\{ \int_{z_{k-1}}^{z_k} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} zdz + \int_{h_{k-1}}^{h_k} \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} k_x \\ k_y \\ k_{xy} \end{bmatrix} z^2 dz \right\}$$

.... (3.1.10)

Since the middle surface strains and curvatures (the ε^0's and κ's) are not a function of z (because these values are always at the middle surface z = 0), they need not to included in the integration. Also, the laminate stiffness matrix is constant for a given ply, so it too will be a constant over the integration of lamina thickness. Then the equations (3.1.9) and (3.1.10) become

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} = \sum_{k=1}^{n} \left\{ \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} \int_{z_{k-1}}^{z_k} dz + \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \int_{z_{k-1}}^{z_k} zdz \right\}$$

.... (3.1.11)

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \sum_{k=1}^{n} \left\{ \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{bmatrix} \int_{z_{k-1}}^{z_k} zdz + \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} \end{bmatrix}_k \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \int_{z_{k-1}}^{z_k} z^2 dz \right\}$$

.... (3.1.12)

Since the middle surface strains and curvatures are not a part of the summations, the laminate stiffness matrix and the h_k terms can be combined to form new matrices defined as:

$$\left.\begin{aligned} A_{ij} &= \sum_{k=1}^{n}\left[\bar{Q}_{ij}\right]_k (z_k - z_{k-1}) \\ B_{ij} &= \frac{1}{2}\sum_{k=1}^{n}\left[\bar{Q}_{ij}\right]_k (z_k^2 - z_{k-1}^2) \\ D_{ij} &= \frac{1}{3}\sum_{k=1}^{n}\left[\bar{Q}_{ij}\right]_k (z_k^3 - z_{k-1}^3) \end{aligned}\right\} \quad \ldots\ldots\ldots (3.1.15)$$

Hence the constitutive equations may be written in matrix form as:

$$\begin{bmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} \quad \ldots\ldots (3.1.16)$$

The equation (3.1.16) may be written in contracted from as:

$$\begin{bmatrix} N \\ M \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} \varepsilon^0 \\ \kappa \end{bmatrix} \quad \ldots\ldots\ldots (3.1.17)$$

The elements of matrices [A], [B], and [D] are termed as extension stiffnesses, coupling stiffnesses and bending stiffnesses respectively

3.2 Free vibrations of laminated composites

The equations of motion for free undamped vibration of laminated composites may be expressed by using the Hamilton principle as

$$[M]\{\ddot{u}\} + [K]\{u\} = \{0\} \quad \ldots\ldots\ldots (3.2.1)$$

where [M] and [K] are system mass and stiffness matrices respectively, and $\{\ddot{u}\}$ and {u} are the acceleration and displacement vectors.

Assuming a harmonic motion, the natural frequencies and the modes of vibration are obtained by solving the generalized eigenvalue problem

$$\{[K] - \omega^2[M]\}X = [0] \quad \ldots\ldots\ldots (3.2.2)$$

where ω is the natural frequency and **X** the mode of vibration.

For a square laminate of side ‘a’ and thickness ‘h’, the eigenvalues are expressed in terms of the non-dimensional frequency parameter $\overline{\omega}$ defined as

$$\overline{\omega} = \frac{\omega a^2}{\pi^2}\sqrt{\frac{\rho h}{D_o}} \quad \ldots\ldots\ldots (3.2.3)$$

where $D_0 = E_{22}h^3/12(1-\nu_{12}\nu_{21})$, ρ is the material density, E is the modulus of elasticity and ν the Poisson’s coefficient.

Also, a shear correction factor (k) is used for different boundary conditions. The value of ‘k’ is taken as 0.8601 for CCCC and CCCF plates, while for SCSC plates, k = 0.822 is used and for SSSS plates, k = 5/6 is considered. Here S stands for simply supported, C for clamped, and F for free boundary conditions.

4. Finite element formulation of laminated composites

The total potential energy consists of three contributions associated, respectively, with in-plane strain energy, shear strain energy and external forces i.e.

$$U = U_p + U_s + U_f \quad \ldots\ldots\ldots (4.1)$$

The in-plane strain energy (U_p) can be divided into a membrane contribution and a bending contribution, and is given by

$$U_p = \frac{1}{2}\int_V \varepsilon_p^T \sigma_p \, dV = \frac{1}{2}\int_{-h/2}^{h/2} \varepsilon_p^T \bar{Q} \varepsilon_p \, dz dA$$

$$= \frac{1}{2}\int_A \left((\varepsilon^0)^T A\varepsilon^0 + 2(\varepsilon^0)^T B\kappa + \kappa^T D\kappa\right) dA$$

$$= U_m + U_{mb} + U_b \quad \ldots\ldots\ldots (4.2)$$

where U_m and U_b are the potential membrane and bending strain energy, respectively. U_{mb} gives the potential strain energy due to the coupling terms between the membrane and bending contributions.

The stiffness matrix is now written as

$$K^{(e)} = K_{mm}^{(e)} + K_{mf}^{(e)} + K_{fm}^{(e)} + K_{ff}^{(e)} + K_{cc}^{(e)}$$

where $K_{mm}^{(e)}$ is the membrane part of the stiffness matrix, $K_{mf}^{(e)}$, $K_{fm}^{(e)}$ are the membrane-bending coupling components, $K_{ff}^{(e)}$ is the bending part, and $K_{cc}^{(e)}$ is the shear part.

The kinetic energy of the plate is given by

$$T^e = \frac{1}{2}\int_A \rho[h\dot{w}^2 + \frac{h^3}{12}\dot{\theta}_x^2 + \frac{h^3}{12}\dot{\theta}_y^2]dA \quad \ldots (4.3)$$

Using the expression for kinetic energy, the mass matrix can be computed by the relation

$$M^e = \int_A \rho N^T \begin{bmatrix} h & 0 & 0 & 0 & 0 \\ 0 & h & 0 & 0 & 0 \\ 0 & 0 & h & 0 & 0 \\ 0 & 0 & 0 & h^3/12 & 0 \\ 0 & 0 & 0 & 0 & h^3/12 \end{bmatrix} NdA$$

in which

$$[N] = \sum_{i=1}^{n} \begin{bmatrix} N_i & 0 & 0 & 0 & 0 \\ 0 & N_i & 0 & 0 & 0 \\ 0 & 0 & N_i & 0 & 0 \\ 0 & 0 & 0 & N_i & 0 \\ 0 & 0 & 0 & 0 & N_i \end{bmatrix}$$

where ‘n’ is the number of nodes considered in the element.

Using the expression for kinetic energy, the mass matrix can be computed by the relation

$$M^e = \int_A \rho N^T \begin{bmatrix} h & 0 & 0 & 0 & 0 \\ 0 & h & 0 & 0 & 0 \\ 0 & 0 & h & 0 & 0 \\ 0 & 0 & 0 & h^3/12 & 0 \\ 0 & 0 & 0 & 0 & h^3/12 \end{bmatrix} NdA$$

in which

$$[N] = \sum_{i=1}^{n} \begin{bmatrix} N_i & 0 & 0 & 0 & 0 \\ 0 & N_i & 0 & 0 & 0 \\ 0 & 0 & N_i & 0 & 0 \\ 0 & 0 & 0 & N_i & 0 \\ 0 & 0 & 0 & 0 & N_i \end{bmatrix}$$

where ‘n’ is the number of nodes considered in the element.

4.3 Finite Element Modelling

A square laminated composite plate of side ‘a’ is considered for analysis using finite element method. The Laminate consists of ‘N’ number of laminas. The laminas have either 0 degree or 90 degree orientation with respect to the material coordinates, i.e., the lamina’s are cross-plied.

Here, the laminated square plate is considered as the domain. The domain is discretized in to sub-domains/finite elements using 8-noded isoparametric quadratic element (serendipity element) as shown in fig. 4.1.

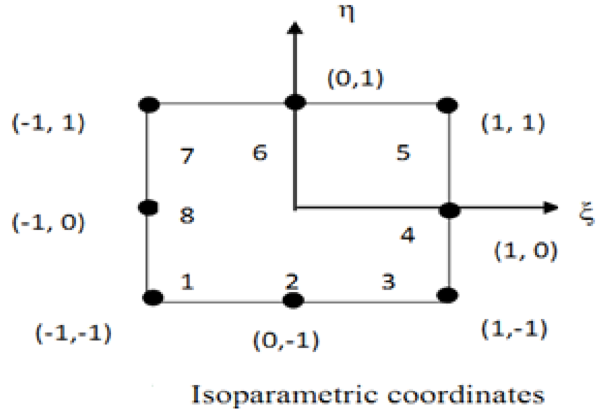

Fig. 4.1: Isoparametric quadratic element

When the coordinates of the element in local coordinates (x,y) are mapped into natural coordinates (ξ, η), then the nodal shape functions are written as

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta)(-\xi-\eta-1)$$

$$N_2 = \frac{1}{2}(1-\xi^2)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1-\eta)(\xi-\eta-1)$$

$$N_4 = \frac{1}{2}(1+\xi)(1-\eta^2)$$

$$N_5 = \frac{1}{4}(1+\xi)(1+\eta)(\xi+\eta-1)$$

$$N_6 = \frac{1}{2}(1-\xi^2)(1+\eta)$$

$$N_7 = \frac{1}{4}(1-\xi)(1+\eta)(-\xi+\eta-1)$$

$$N_8 = \frac{1}{2}(1-\xi)(1-\eta^2)$$

The integrals in the stiffness and mass matrices are evaluated numerically using the Gauss quadrature rule, in the limits of -1 to +1, i.e.

$$K_{ij} = \int_{-1}^{1}\int_{-1}^{1} B_i^T D B_j |J|\, d\xi d\eta$$

where $|J|$ is the determinant of Jacobian matrix expressed as

4.4 Solution procedure

The steps followed in the analysis are arranged in sequence as under :

Step I: Select the material properties (viz. E, ν, G, ρ) for the lamina materials.

Step II: Generate the mesh i.e. decide upon the number and size of finite elements along x and y directions. Here, 8-noded isoparametric quadratic elements (serendipity elements) have been employed for forming the mesh.

Step III: Calculate the nodal shape functions and their derivatives at all the nodes of each element.

Step IV: Generate element mass and stiffness matrices for each element.

Step V: Assemble the element matrices to global matrix following the nodal connectivity.

Step VI: Apply the boundary conditions.

Step VII: Solve the global equation of motion for eigen values. The equation of motion for the free vibration of a plate is of the form

$$[M]\{\ddot{u}\} + [K]\{u\} = \{0\}$$

$$\text{or} \qquad \{[K] - \omega^2[M]\}X = [0]$$

The eigen values indicate the natural frequency of vibration of the composite plates.

Step VIII: Perform the convergence criterion by increasing the number of elements in x- and y-directions.

5. RESULTS AND DISCUSSIONS

The numerical results are obtained for two types of materials with the following elastic properties:

Material I : $E_1/E_2 = 40$; $G_{12} = G_{13} = 0.6\, E_2$; $G_{23} = 0.5\, E_2$; $\nu_{12} = 0.25$.

Material II : $E_1/E_2 = 25$; $G_{12} = G_{13} = 0.5\, E_2$; $G_{23} = 0.2\, E_2$; $\nu_{12} = 0.25$.

Table 5.1: Fundamental frequency parameters of a 3-ply symmetric cross-ply ($90^o/0^o/90^o$) square laminate with SSSS boundary conditions for Material I

a/h ratio	*Fundamental frequency parameter*	
	Present	*Ref. [5]*
5	*10.2190*	*10.264*
10	*14.7670*	*14.702*
20	*17.5235*	*17.483*
50	*18.6509*	*18.641*
100	*18.8317*	*18.828*

Table 5.1 shows the values of fundamental frequency parameters $[\bar{\omega} = \frac{\omega a^2}{\pi^2}\sqrt{\frac{\rho h}{D_o}} = (\frac{\omega a^2}{h})\sqrt{\frac{\rho}{E_2}}]$ of a symmetric cross-ply square laminate having orientation ($90^o/0^o/90^o$) with all edges simply supported (SSSS) for material I corresponding to different values of side to thickness ratio (a/h). The values of frequency parameter obtained are compared with those reported by Hadian and Nayfeh [5]. It is observed from the comparison in table that the present results are in very good agreement with the theoretical results of ref. [5].

The variation of fundamental frequency parameter with side to thickness ratio (a/h) for symmetric cross-ply ($90^o/0^o/90^o$) laminated composite square plate having all edges simply supported (SSSS) for material I is plotted in fig. 5.1. The results by Hadian and Nayfeh [5] are also plotted in the figure.

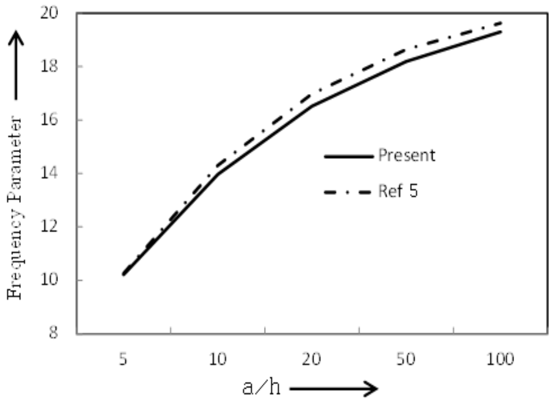

Fig. 5.1 Variation of fundamental frequency parameters with a/h ratio for symmetric cross-ply ($90^o/0^o/90^o$) laminated square plate of material I with SSSS boundary condition

6. Conclusion

Finite element analysis of symmetric cross-ply laminated composite square plate is carried out, using a 8–noded isoparametric quadratic element to predict the fundamental natural frequency of free vibration under different sets of boundary conditions. The present model is developed based on the First order Shear Deformation Theory (FSDT). This theory uses a shear correction factor to approximate the transverse shear stresses. A computer program is written in MATLAB to get various results. The accuracy of results obtained using the present formulation is demonstrated by comparing the results with existing literatures. It is concluded that the frequency parameter increases with increase in the side to thickness ratio for any set of boundary conditions at edges of the laminate. Also, the effect of various boundary conditions at edges of laminate has been analyzed. Various results of natural frequency are also obtained by the FEA software ANSYS.

References

[1] Bhat R.B., “Natural frequencies of rectangular plates using characteristic orthogonal polynomials in Rayleigh–Ritz method”; Journal of sound and vibration, (1985); 102(4); pp. 493–499.

[2] Dickinson S.M., Blasio A., “On the use of orthogonal polynomials in the Rayleigh-Ritz method for the study of the flexural vibration and buckling of isotropic and orthotropic rectangular plates”; Journal of sound and vibrations, (1986); 108(1); pp. 51–62.

[3] Leissa A.W., Narita Y., “Vibration studies for simply supported symmetrically laminated rectangular plates”; Composite Structures, (1989);12; pp.113–132.

[4] Soldatos K.P., Timarci T., “A unified formulation of laminated composite, shear deformable, five-degrees-of-freedom cylindrical shell theories”; Composite Structures, (1993); 25; pp. 165–171.

[5] Hadian J., Nayfeh A.H., “Free vibration and buckling of shear-deformable cross-ply laminated plates using the state-space concept”; Computers and Structures, (1993); Vol. 48, No. 4; pp. 677–693.

[6] Brown R.E., Stone M.A., “On the use of polynomial series with the Rayleigh–Ritz Method”; Composite Structures, (1997); Vol. 39, No. 3–4; pp.191–196.

[7] Messina A., Soldatos K.P., “Influence of edge boundary conditions on the freevibrations of cross-ply laminated circular cylindrical panels”; J. Acoust. Soc. Am., (1999); Vol. 106, No. 5; pp. 2608–2620.

[8] Messina A., Soldatos K.P., “Vibration of completely free composite plates and cylindrical shell panels by a higher-order theory”; International Journal of Mechanical Sciences, (1999); 41; pp. 891–918.

