Relation between Torsion and Normal Curvature of a geodesic on a developable

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Abstract- A developable surface is the surface generated by one-parameter family of planes. As there are three planes namely the osculating plane, the normal plane and the rectifying plane of a moving trihedral on any given space curve and equation of these planes contain only one parameter usually the arc length 's' so envelopes of these planes are developable surfaces. A geodesic on a developable surface (rectifying developable) is the line of shortest distance on the developable. In this paper, a relation between torsion and curvature of a geodesic on a developable surface is obtained mainly using Euler's Theorem and some other basic concepts of differential geometry.

Keywords*:* Developable, Torsion, Curvature, Rectifying plane, arc length.

I. INTRODUCTION

A developable is the surface which is formed by bending a planar surface without tearing or stretching it. It is type of ruled surface which has common tangent plane at all the points along a generator [1] .A developable surface is the surface generated by one-parameter family of planes. As there are three planes namely the osculating plane, the normal plane and the rectifying plane of a moving trihedral on any given space curve and equation of these planes contain only one parameter usually the arc length 's' so envelopes of these planes are developable surfaces. A geodesic on a developable surface (rectifying developable) is the line of shortest distance on the developable.

Differential Geometry Properties of a developable surface

Following are some important differential geometry properties of a developable:

1. It can be mapped onto a plane isometrically.

2. At corresponding points on isometric surfaces, Gaussian curvature is same.

3. On isometric surfaces, the corresponding curves have some geodesic curvature at corresponding points.

4. A geodesic on a developable maps to a line in a plane. [2]

Theorem: The Gaussian Curvature of a developable surface is zero everywhere. [3]

II. CURVATURE OF NORMAL SECTION OF A DEVELOPABLE

The unit tangent vector **t** and unit normal vector **n** of a curve C at a point P on surface S are related by

$$\mathbf{k} = \mathbf{k_n} + \mathbf{k_g}$$

where $\mathbf{k_n}$ is normal curvature vector and $\mathbf{k_g}$ is geodesic curvature vector. The normal curvature vector $\mathbf{k_n}$ is component of curvature vector **k** in the normal direction to curve C at point P. In terms of fundamental magnitudes, it is given by

$$\mathbf{k_n} = (Ldu^2+2M\,dudv+Ndv^2)/(Edu^2+2F\,dudv+Gdv^2)$$

The extreme values of $\mathbf{k_n}$ are called principal curvatures [2] and are given by

$$(EG-F^2)\,\mathbf{k_n}^2-(EN+LG-2FM)\,\mathbf{k_n} + (LN-M^2) = 0$$

$$\text{or}\quad H^2\,\mathbf{k_n}^2 - (EN+LG-2FM)\,\mathbf{k_n} + T^2 = 0$$

where $H^2= EG-F^2$ and $T^2=LN-M^2$

Let us set

$$K = T^2/H^2 \text{ and } B = (EN + LG- 2FM)/2H^2$$

The above equation reduces to

$$\mathbf{k_n}^2 - 2B\,\mathbf{k_n} + K= 0$$

The quantities K and B are called Gaussian curvature and mean curvature respectively.
On solving the above equations for maximum and minimum values, we get

$$\mathbf{k_{max}} = B + (B^2 - K)^{1/2}$$

And $$\mathbf{k_{min}} = B - (B^2 - K)^{1/2}$$

Theorem : At least one of the principal Curvatures is zero at every point on a developable surface.

Proof: Since Gaussian Curvature of a developable is zero everywhere so

$$\mathbf{k_{max}} = B + (B^2 - K)^{1/2} = B + |B| \quad \text{and}$$

$$\mathbf{k_{min}} = B - (B^2 - K)^{1/2} = B - |B|$$

If $B > 0$, $\mathbf{k_{max}} = 2B$, $\mathbf{k_{min}} = 0$

If $B = 0$, $\mathbf{k_{max}} = 0$, $\mathbf{k_{min}} = 0$

If $B < 0$, $\mathbf{k_{max}} = 0$, $\mathbf{k_{min}} = 2B$

So at least one of the principal curvature is zero at every point on a developable surface.

Euler's Theorem: Euler's Theorem relates the normal curvature at a point on the surface with principal curvatures i.e. directions having least and most curvature. [4] If ψ is the angle which the direction (du, dv) of normal section makes with the principal direction dv =0 then

$$\mathbf{k_n} = \mathbf{k_a} \cos^2\psi + \mathbf{k_b} \sin^2\psi$$

where $\mathbf{k_a}$ and $\mathbf{k_b}$ are principal curvatures.

Definition
A Geodesic on a surface is a curve whose osculating plane at each point contains the normal to surface at that point. [5] A Geodesic is a generalization of the concept of a straight line to curved surfaces. On any surface, Geodesics are special intrinsic curves. The problem is, given any two points A and B on the surface, to find, out of all the arcs joining A and B, those which give the least arc length. A Geodesic may be regarded as curves of stationary rather than strictly shortest distance on the surface. [6]

Theorem: The necessary and sufficient condition that a curve other than the straight line surface be geodesic is that the surface be the rectifying developable of the curve. [7]

Theorem: The torsion of the geodesic tangent at any point of a curve on a surface is given by

$$\tau = (\mathbf{k_b} - \mathbf{k_a}) \sin\psi \cos\psi$$

where ψ is the angle between the tangent and any of the principal directions.

Proof: Let us take principal directions as parametric curves.
For parametric Curves, F=0, M=0 and

$$\tau = \{(EM-FL)\, u'^2 + (EN-GL)\, u'v' + (FN-GM)\, v'^2\} / H$$

So $$\boldsymbol{\tau} = (EN-GL)u'v' / H$$

$$= (EG / \sqrt{EG})\, (N/G - L/E)\, u'v'$$

$$= \sqrt{EG}\, (N/G - L/E)\, u'v'$$

Also if ψ is the angle between the geodesic tangent and parametric curves v= constant then

$$\cos\psi = E\, ll' + F\, (lm' + ml') + G mm'$$

$$\text{and} \quad \sin\psi = H\, (lm' - ml')$$

Now $l = 1/\sqrt{E}$, $m=0$, $l' = du/ds$, $m' = du/ds$

$\cos\psi = \sqrt{E}\, u'$ and $\sin\psi = \sqrt{G} v'$

So $u' = \cos\psi/\sqrt{E}$, $v' = \sin\psi/\sqrt{E}$

Further, the principal curvatures are given by

$$\mathbf{k_a} = L/E, \quad \mathbf{k_b} = N/G$$

$$\tau = \sqrt{EG}\, (\mathbf{k_a} - \mathbf{k_b})\, (\cos\psi / \sqrt{E})\, (\sin\psi/\sqrt{G})$$

$$\text{i.e. } \tau = (\mathbf{k_b} - \mathbf{k_a}) \sin\psi \cos\psi$$

III. RELATION BETWEEN TORSION AND CURVATURE OF GEODESIC ON A DEVELOPABLE

On using the above results

$$\mathbf{k_n} - \mathbf{k_a} = \mathbf{k_a} \cos^2\psi + \mathbf{k_b} \sin^2\psi - \mathbf{k_a}$$

$$= (\mathbf{k_b} - \mathbf{k_a}) \sin^2\psi$$

And

$$\mathbf{k_a} - \mathbf{k_n} = \mathbf{k_b} - \mathbf{k_a} \cos^2\psi - \mathbf{k_b} \sin^2\psi$$

$$= (\mathbf{k_b}-\mathbf{k_a}) \cos^2 \psi$$

$$\text{So } (\mathbf{k_n} - \mathbf{k_a})(\mathbf{k_b} - \mathbf{k_n}) = (\mathbf{k_b}-\mathbf{k_a})^2 \sin^2 \psi \cos^2 \psi$$

$$= \tau^2$$

For developable surface,

$$\text{let } \mathbf{k_a} = 0$$

$$\tau^2 = \mathbf{k_n} (\mathbf{k_b} - \mathbf{k_n})$$

Also, by Euler's Theorem

$$\mathbf{k_n} = \mathbf{k_b} \sin^2 \psi$$

$$\text{or } \mathbf{k_b} = \mathbf{k_n} \operatorname{cosec}^2 \psi$$

$$\text{So } \tau^2 = \mathbf{k_n} (\mathbf{k_n} \operatorname{cosec}^2 \psi - \mathbf{k_n})$$

$$\text{or } \tau^2 = \mathbf{k_n}^2 (\operatorname{cosec}^2 \psi - 1)$$

$$\text{or } \tau^2 = \mathbf{k_n}^2 \cos^2 \psi$$

$$\text{or } \mathbf{k_n} = \tau \tan \psi$$

i.e. normal curvature of a geodesic on a developable is the product of torsion and tangent of angle which the geodesic tangent makes with principal direction.

IV. CONCLUSION

The normal curvature of a geodesic on a developable is the product of torsion of the geodesic and tangent of the angle which the direction (du, dv) of normal section makes with principal direction dv=0.

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