A Study of Variation of the Probability of M/D/c Deterministic Queueing Model

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Abstract **— Study a variation of the probability of M/D/c queueing model in which service time of customers is deterministic. The number of server channel modified depending on the length of queue. Specifically, apart from the regular number of server, we can increase the number of server. The model represents situations such as when the system requires a new server depending on the length of the queue can be provided, so that the waiting time can be reduce.**

Keywords **— Queueing models FCFS; M/D/c; Poisson distribution; Queue; Utilization factor; Waiting time.**

I. INTRODUCTION

The mathematical study of waiting lines is known as Queueing Theory. Queueing Theory was originally developed mostly in the context of telephone traffic engineering but it has found applications in several disciplines such as engineering, operation research, and computer science, with practical applications in such areas as layout of manufacturing systems, airport traffic modeling, measurement of computer performances, analysis of traffic control, study of telecommunications systems and even to model decision-making to replace a goalie in a hockey game. The earliest mention of Queueing Theory was made in 1909 in a paper by A.K. Erlang. In 1951 David G. Kendall provided a systematic treatment of the study of basic queues and included in his paper the first mention ever of the term “queueing systems”. Later in 1953, Kendall also introduced a formal classification of queueing systems. Since then, various queueing models and their analyses have occupied a voluminous part of the operations research literature.

II. THE NOTATION

As mentioned earlier, Queueing Theory has been successfully used to model, analyze, and solve complex systems, using analytical, numerical, and simulation techniques [1]. A basic queueing system is specified by identifying the essential components that make up such a system- arrival process, service process, number of servers, buffer size to hold waiting entities, size of the calling population, and service priority. A system is indicated in a notational form A/S/N/C/P/D. The arrival process, A, is denoted by specifying the 6 distributions of inter-arrival times. For example, if the inter-arrival times are exponentially distributed, the letter M is used. This is due to the Markovian, or memory-less property of the exponential distribution. If the inter-arrival times are assumed to be independent and have an arbitrary, general distribution, the notation of GI is used. Another common interarrival distribution is an Erlang distribution of the order k. This is the distribution of the sum of k independent and identically distributed (i.i.d) exponential random variables. This distribution is denoted by the symbol Ek. A generalization of the Erlang distribution in which the inter-arrival times are associated with the times of absorption in a finite-state Markov process with one absorbing state, is known as the phase-type distribution [10]. The notation used to indicate such a distribution is PH. various other inter-arrival time distributions have been used in the literature to model specific queueing systems. The second parameter S stands for the service process and is indicated in a way similar to the arrival process described above. The parameter N indicates the number of servers, C represents the maximum system capacity, the population size is denoted by P and D stands for the service discipline such as first in first out, last in first out, random. By default, system capacity and population size are taken to be infinite and the service is process is assumed to be first-in-first out. A good review of a variety of queueing models can be found in [1] and [7].

III. QUEUING MODELS AND KENDALL’S NOTATION

In most cases, queuing models can be characterized by the following factors:

A. *Arrival time distribution*. Inter-arrival times most commonly fall into one of the following distribution patterns: a Poisson distribution, a Deterministic distribution, or a General distribution. However, inter-arrival times are most often assumed to be independent and memoryless, which is the attributes of a Poisson distribution.

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B. *Service time distribution*. The service time distribution can be constant, exponential, hyperexponential, hypo-exponential or general. The service time is independent of the inter-arrival time.

C. *Number of servers*. The queuing calculations change depends on whether there is a single server or multiple servers for the queue. A single server queue has one server for the queue. This is the situation normally found in a grocery store where there is a line for each cashier. A multiple server queue corresponds to the situation in a bank in which a single line waits for the first of several tellers to become available.

D. *Queue Lengths (optional)*. The queue in a system can be modeled as having infinite or finite queue length.

E. *System capacity (optional)*. The maximum number of customers in a system can be from 1 up to infinity. This includes the customers waiting in the queue.

F. *Queuing discipline (optional)*. There are several possibilities in terms of the sequence of customers to be served such as FIFO (First In First Out, i.e. in order of arrival), random order, LIFO (Last In First Out, i.e. the last one to come will be the first to be served), or priorities. Kendall, in 1953, proposed a notation system to represent the six characteristics discussed above. The notation of a queue is written as:

A/B/P/Q/R/Z

Where, A, B, P, Q, R and Z describe the queuing system properties.

A describes the distribution type of the inter arrival times.
B describes the distribution type of the service times.
P describes the number of servers in the system.
Q (optional) describes the maximum length of the queue.
R (optional) describes the size of the system population.
Z (optional) describes the queuing discipline.

IV. THE MATHEMATICAL MODEL:M/D/C : FCFS/∞/∞

In the last decade sevral useful approximations has been obtained for the average wating time I M/G/C queue , see[4]& [8] for further discussions. This is a model with poisons arrival, determinstic service with multichannel, first come first serve discipline and infinite polpulation. The state probability for the continuous M/D/c queue with infinite buffer can be obtained [5]. In this model to serve one unit more than one server are available, the arrival of one unit is depend on markovin process and service time is determinstic. Although several approximation for he average queue length in M/D/c queue can be fount in [5].

Here n: the total no. of customer in the system at time t.

c: Number of server channel (1≤c)

Case I: If n ≤ c, In this case no customer have to wait for the service, but there is a possibility that some server may be idle.

Case II: n > c, In this case, at any time t, c customer will be serve and (n-c) customers willl be waiting in the queue. If λ and μ be the parameter of I/P, O/P and μ (= k (say)) is constant service discipline.

Probability of n arrival in time t = $\frac{e^{-\lambda t}(\lambda t)^n}{n!}$

$$\lambda_n = \lambda_0 , \quad \mu_n = \mu_n \quad if\ n = 0$$
$$\lambda_n = n\lambda , \quad \mu_n = n\mu \quad if\ n \le c$$
$$\lambda_n = n\lambda , \quad \mu_n = c\mu \quad if\ n \ge c$$

First we find $P_n(t + dt)$ for n ≤ c. There are following three ways are possible

Table 1

Event	No. of units at time t	No. of arrivals in time dt	No. of service in time dt	No. of units at time t + dt
1	n	0	0	n
2	n-1	1	0	n
3	n+1	0	1	n

Probability of Event 1

$$= P_n(t)(1 - \lambda_n dt)(1 - \mu_n dt)$$
$$= P_n(t)(1 - n\lambda dt)(1 - n\mu dt)$$
$$= P_n(t)(1 - n\lambda dt)(1 - nkdt)$$
$$= P_n(t)[1 - (n\lambda + nk)dt]$$

Probability of Event 2

$$= P_{n-1}(t)[(\lambda_{n-1}dt)][(1 - \mu_{n-1}dt)]$$
$$= P_{n-1}(t)[(n-1)\lambda dt][(1 - (n-1)\mu dt)]$$
$$= P_{n-1}(t)[(n-1)\lambda dt][(1 - (n-1)kdt)$$
$$= P_{n-1}(t)[(n-1)\lambda dt]$$

Probability of Event 3

$$= P_{n+1}(t)[(1 - \lambda_{n+1}dt)][(\mu_{n+1}dt)]$$
$$= P_{n+1}(t)[(1 - (n+1)\lambda dt)][(n+1)\mu dt)]$$
$$= P_{n+1}(t)[(1 - (n+1)\lambda dt)][(n+1)kdt)]]$$
$$= P_{n+1}(t)[(n+1)kdt]$$

Therefore

$$P_n(t + dt) = P_n(t)[1 - (\lambda + k)ndt] + P_{n-1}(t)[(n-1)\lambda dt] + P_{n+1}(t)[(n+1)kdt] \quad \ldots\ldots (1)$$

Set $dt \longrightarrow 0$

$\frac{d}{dt}P_n(t) = P_n(t)[-(\lambda + k)n] + P_{n-1}(t)[(n-1)\lambda] + P_{n+1}(t)[(n+1)k]$

Similirly Now we find $P_n(t+dt)$ for c < n.

Probability of Event 1

$$= P_n(t)[(1-\lambda_n dt)][(1-ckdt)]$$
$$= P_n(t)[1-n\lambda dt - ckdt]$$
$$= P_n(t)[1-(n\lambda + ck)dt]$$

Probability of Event 2

$$= P_{n-1}(t)[(\lambda_{n-1}dt)][(1-\mu_{n-1}dt)]$$
$$= P_{n-1}(t)[(n-1)\lambda dt][(1-ckdt)]$$
$$= P_{n-1}(t)[(n-1)\lambda dt]$$

Probability of Event 3

$$= P_{n+1}(t)[(1-\lambda_{n+1}dt)][(\mu_{n+1}dt)]$$
$$= P_{n+1}(t)[(1-(n+1)\lambda dt)][(ckdt)]$$
$$= P_{n+1}(t)[(1-(n+1)\lambda dt)][(n+1)kdt)]]$$
$$= P_{n+1}(t)[ckdt]$$

Therefore

$$P_n(t+dt) = P_n(t)[1-(n\lambda + ck)dt] + P_{n-1}(t)[(n-1)\lambda dt] + P_{n+1}(t)[ckdt]$$

………………… (2)

Set $dt \longrightarrow 0$

$\frac{d}{dt}P_n(t) = P_n(t)[-(n\lambda + ck)] + P_{n-1}(t)[(n-1)\lambda] + P_{n+1}(t)[ck]$

$P_0(t+dt) = P_0(t)[1-ckdt] + P_1(t)[ckdt]$

$dt \longrightarrow 0$

$\frac{d}{dt}P_0(t) = P_0(t)[-ck] + P_1(t)[ck]$

.... (3)

For steady state by equation 1, 2 & 3

$P_0(t)ck = P_1(t)ck$

$\Longrightarrow P_0 = P_1$; $if\ n = 0$ .... (4.1)

$P_n(\lambda + k)n = P_{n-1}(n-1)\lambda + P_{n+1}(n+1)k$; $if\ n \le c$ …..….. (4.2)

$P_n(n\lambda + ck) = P_{n-1}(n-1)\lambda + P_{n+1}(t)ck$; $if\ c < n$ } ...…….. (4.3)

On simplification of equation (4)

$P_0 = P_1$; $if\ n = 0$ …………………. (5.1)

$P_n\lambda n = P_{n+1}(n+1)k$; $if\ n \le c$ ……………………… (5.2)

$P_n n\lambda = P_{n+1}ck$; $if\ c < n$ ……………………… (5.3)

To find steady state from equation (5)

$Put\ n = 1$

$P_0 = P_1$; $if\ n = 0$

$1\lambda P_1 = P_2(2k)$; $if\ n \le c$

$1\lambda P_1 = P_2(ck)$; $if\ c < n$

$Put\ n = 2$

$2\lambda P_2 = P_3(3k)$; $if\ n \le c$

$2\lambda P_2 = P_3(ck)$; $if\ c < n$

$Put\ n = 3$

$3\lambda P_3 = P_4(4k)$; $if\ n \le c$

$3\lambda P_3 = P_4(ck)$; $if\ c < n$

Continue this process

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In general

$P_{n+1} = \frac{n}{n+1}\frac{\lambda}{\mu}P_n$; $if\ n \le c$ ......……. (6)

$P_{n+1} = \frac{n}{c}\frac{\lambda}{k}P_n$; $if\ c < n$ ………… (7)

For n ≤ c

$P_0 = P_1 \Longrightarrow P_1 = P_0$

$1\lambda P_1 = P_2(2k) \Longrightarrow P_2 = \frac{1}{2}\frac{\lambda}{k}P_0$

$2\lambda P_2 = P_3(3k) \Longrightarrow P_3 = \frac{1}{3}\left(\frac{\lambda}{k}\right)^2 P_0$

$3\lambda P_3 = P_4(4k) \Longrightarrow P_4 = \frac{1}{4}\left(\frac{\lambda}{k}\right)^3 P_0$

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………… $\Longrightarrow P_n = \frac{1}{n}\left(\frac{\lambda}{k}\right)^{n-1} P_0$ ....……..(8)

For n > c

$P_0 = P_1 \Longrightarrow P_1 = P_0$

$1\lambda P_1 = P_2(ck) \Longrightarrow P_2 = \frac{1}{c}\frac{\lambda}{k}P_0$

$2\lambda P_2 = P_3(ck) \Longrightarrow P_3 = \frac{1.2}{c^2}\left(\frac{\lambda}{k}\right)^2 P_0$

$3\lambda P_3 = P_4(ck) \Longrightarrow P_4 = \frac{1.2.3}{c^3}\left(\frac{\lambda}{k}\right)^3 P_0$

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$$\Rightarrow P_n = \frac{(n-1)!}{c^{n-1}} \left(\frac{\lambda}{k}\right)^{n-1} P_0 \qquad \text{…….....…. (9)}$$

As

$$\sum_{n=0}^{\infty} P_n = 1 \Rightarrow \sum_{n=0}^{c} P_n + \sum_{n=c+1}^{\infty} P_n = 1$$

$$\Rightarrow \sum_{n=0}^{c} \frac{1}{n} \left(\frac{\lambda}{k}\right)^{n-1} P_0 + \sum_{n=c+1}^{\infty} \frac{(n-1)!}{c^{n-1}} \left(\frac{\lambda}{k}\right)^{n-1} P_0 = 1$$

$$\therefore P_0 = \frac{1}{\sum_{n=0}^{c} \frac{1}{n} \left(\frac{\lambda}{k}\right)^{n-1} P_0 + \sum_{n=c+1}^{\infty} \frac{(n-1)!}{c^{n-1}} \left(\frac{\lambda}{k}\right)^{n-1} P_0} \qquad \text{………… (10)}$$

We get the following solution:

$$P_n = \frac{1}{n} \left(\frac{\lambda}{k}\right)^{n-1} P_0 \qquad For\ n \leq c \qquad \text{……….. (11.1)}$$

$$P_n = \frac{(n-1)!}{c^{n-1}} \left(\frac{\lambda}{k}\right)^{n-1} P_0 \qquad For\ n > c \qquad \text{…........ (11.2)}$$

Where P_0 is given by equation (10).

V. CONCLUSION

In this paper we studied a variation of the probability of M/D/c system with deterministic service time, with poissons arrival and several service channels. The results explain the complex nature of the probability depend up on the number of customers. We can determine the probability in both the cases wlile the number of customers are less than equal to or greater than the available number of servers.

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