On The Unbiased Windmill Graphs

S.Kalyana kumar[1], Dr.Mata Ambar[2]
[1]Research Scholar/Mathematics, University of Allahabad, Allahabad
[2]Professor, Department of Mathematics, University of Allahabad, Allahabad.

ABSTRACT
Let G be a graph with vertex set V(G) and edge set E(G), and let A ={0,1}. A labeling f : V(G) $\rightarrow$ A induces an edge partial labeling f^* : E(G) $\rightarrow$ A defined by $f^*(xy) = f(x)$, if and only if f(x)=f(y) for each edge xy $\in$ E(G). For i $\in$ A, let $v_f(i)$ = card{v $\in$ V(G) : f(v) = i} and $e_{f*}(i)$ = card{e $\in$ E(G) : $f^*(e)$ = i}. A labeling f of a graph G is said to be friendly if | $v_f(0) - v_f(1)$ | $\le$ 1. If, | $e_f(0) - e_f(1)$ | $\le$ 1 then G is said to be **balanced**. We also show that here several families of balanced graphs for regular windmill and general windmill graphs.

Key words: vertex labeling, friendly labeling, cordiality, balanced, NP-complete.

1.Introduction.

A labeling problem of graphs which is called cordial graph labeling was introduced by Cahit [2] in 1986. Let G be a graph with vertex set V(G) and edge set E(G). A binary vertex labeling of G is a mapping from V(G) into the set {0,1}. For each vertex labeling f of G, Cahit considered a binary edge labeling f#: E→ {0,1}, defined by f# ({u, v}) = | f(u) – f(v)| for all {u, v} in E(G). Let $V_0^f(G)$ and $V_1^f(G)$ denote the number of elements in V(G) that are labeled by 0 and 1 under the mapping f respectively. Likewise, let $e_0^{f\#}(G)$ and $e_1^{f\#}(G)$ denote the number of elements in E(G) that are labeled by 0 and 1 under the induced function f# respectively. Cahit called a graph **cordial** if it has the following properties:

(i) $|v_0^f(G) - v_1^f(G)| \le 1$ and
(ii) $|e_0^{f\#}(G) - e_1^{f\#}(G)| \le 1$.

Several constructions of cordial graphs, in particular, the Cartesian product, composition of graphs and tensor products, are considered in [1, 7, 11, 12, 14, 16, 17.19,21,22,24,25]. For some new and unsolved problems, the reader refer to [4,7,8].

Lee, Liu and Tan considered another labeling problem, the ***balanced labeling*** problem [20]. For any binary vertex labeling, a partial edge labeling f^* of G can be defined in the following way. For each edge {u, v} in E(G), where u, v $\in$ V(G)

$$f^*(\{u, v\}) = \begin{cases} 0 & \text{if } f(u) = f(v) = 0 \\ 1 & \text{if } ((u) = f(v) = 1 \end{cases}$$

Note that if f(u) $\neq$ f(v), the edge {u, v} is not labeled by f^*. Thus f^* is a partial function from E(G) into the set {0, 1}, and we shall refer f^* as the induced partial function of f. Let e_0^f (G) and e_1^f (G) denote the number of elements in E(G) that are labeled by 0 and 1 under the induced partial function f^* respectively.

Hence,

$v_0^f(G) = |\{u \in V(G) \mid f(u) = 0\}|$
$v_1^f(G) = |\{u \in V(G) \mid f(u) = 1\}|$
$e_0^{f*}(G) = |\{\{u, v\} \in E(G) \mid f^*(\{u, v\}) = 0\}|$
$e_1^{f*}(G) = |\{\{u, v\} \in E(G) \mid f^*(\{u, v\}) = 1\}|$

With this notation, we now introduce the notion of a balanced graph.

Definition 1.1. Let G be a graph. G is a balanced graph, or G is balanced, if there is a binary vertex labeling f of G that satisfies the following conditions:

(i) $|v_0^f(G) - v_1^f(G)| \le 1$ and
(ii) $|e_0^{f*}(G) - e_1^{f*}(G)| \le 1$.

A graph G is said to be **strongly vertex-balanced** if G is a balanced graph and $v_0^f(G) = v_1^f(G)$. Similarly, a graph G is said to be **strongly edge-balanced** if G is a balanced graph and $e_0^{f*}(G) = e_1^{f*}(G)$. If G is both strongly vertex-balanced and strongly edge-balanced, we say that G is **strongly balanced**. (We will omit the superscripts f and f^* when the context is clear).

Example 1. Figure 1 shows that the BI(G) = {0,1,2}.

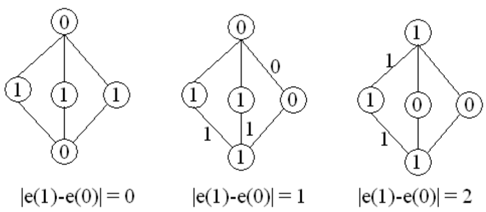


Figure 1.

Example 2. The (6,7)-graph G is balanced

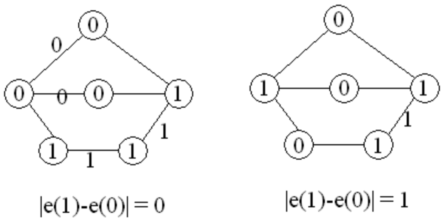


Figure 2.

Example 3.Figure 4 shows that C_3xP_3 is balanced and $C_4x P_2$ is strongly balanced.

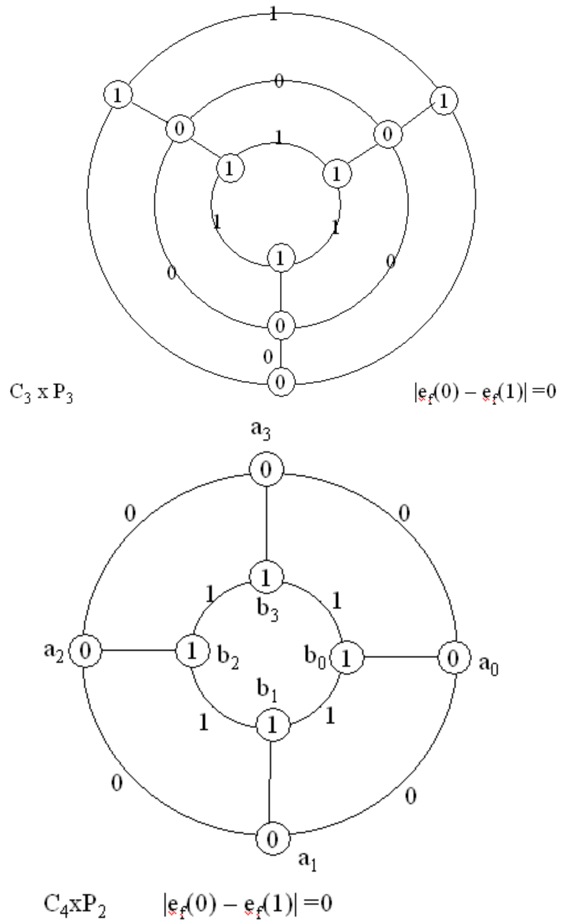


Figure 3.

The following results were established in [20]:

Theorem 1.1 Let G be a k-regular graph with p vertices and q edges,

(i) G is strongly balanced if and only if p is even;
(ii) G is balanced if and only if p is odd and k = 2.

Corollary 1.2 Every cycle C_m is a balanced graph.

Corollary 1.3 For complete graph on n vertices K_m,

(i) K_m is a strongly balanced graph if m is even;
(ii) If m is odd, K_m is balanced if and only if m = 3.

Theorem 1.4 Every path P_m is balanced for $m \geq 1$ and is strongly balanced if m is even.

Theorem 1.5 The complete bipartite graph $K_{m,n}$ is balanced if and only if one of the following conditions holds:

(i) both m and n are even;
(ii) both m and n re odd and $|m-n| \leq 2$;
(iii) one of m and n, say m, is odd, n = 2t and t = -1, 0, or 1 (mod |m-n|).

Let $K(m_1, m_2, \ldots, m_n)$ be the one-point amalgamation of the complete graphs with $m_1, m_2, \ldots, m_n$ vertices. Call the point at which the complete graphs are amalgamated the center of $K(m_1,\ldots m_n)$. If k of the m values are equal to the same value a, and if no confusion could arise, we use a^k to denote these values. Thus $K(a^k)$ is the regular windmill with k component each is a complete graph.

Suppose a graph G has p vertices and q edges. Assume that p_i of the vertices are of degree r_i, for i = 1, 2, …, n, and r_i are integers such that $r_1 < r_2 < \ldots < r_n$.

Let

$$S_v = \sum_{i=1}^{n} (p_i - 2a_i) \qquad (1)$$

$$S_e = \sum_{i=1}^{n} (p_i - 2a_i)\, r_i / 2 \qquad (2)$$

In [13], a necessary and sufficient condition for a graph to be (strongly) balanced is given. .

Theorem 1.6. Let G be a (p, q)-graph with S_v and S_e as defined in (1) and (2) respectively. G is balanced if and only if there exists a set of integers $\{a_i \mid 0 \leq a_i \leq p_i, i = 1, 2, \ldots, n\}$ such that $|S_v| \leq 1$ and $|S_e| \leq 1$. Furthermore, G is strongly balanced if and only if there exists a set of integers $\{a_i \mid 0 \leq a_i \leq p_i, i = 1, 2, \ldots, n\}$ such that $S_v = 0$, and $S_e = 0$.

We illustrate how this result can be used to determine the (strongly) balanced ness of several families of windmill graphs. In [1], we consider the cordialness for windmill graphs. In this paper we investigate which windmill graphs are balanced. For other results of balanced graphs see [10,13].

2. Regular Windmill graphs

Theorem 2.1. The regular windmill graph $K(3^k)$ is balanced if and only if k = 2,3.

Proof.

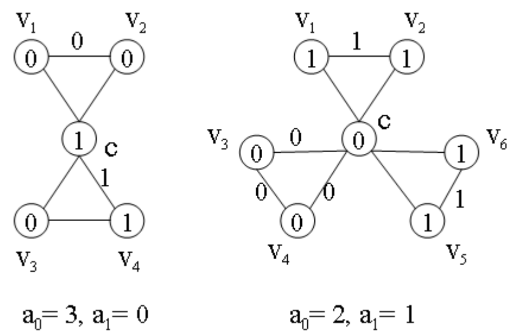


Figure 4.

Theorem 2.2. For $n \geq 4$, the regular windmill graph $K(n^k)$ is balanced if and only if k = 2.

Proof.

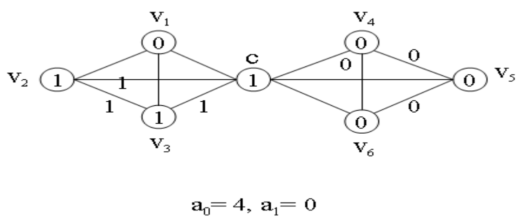


Figure 5.

3. Balanced Windmill graphs with two different types of components.

Theorem 3.1. The windmill graph K(2 ^{k}, 3) is balanced if and only if k = 1,2,3,4.

Proof.

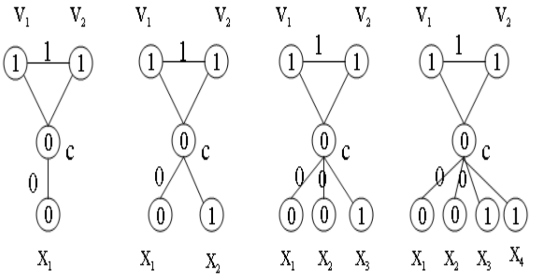


Figure 7.

Theorem 3.1. The windmill graph K(2 ^{k}, 4) is balanced if and only if k = 1,2,3,4,5,6.

Proof.

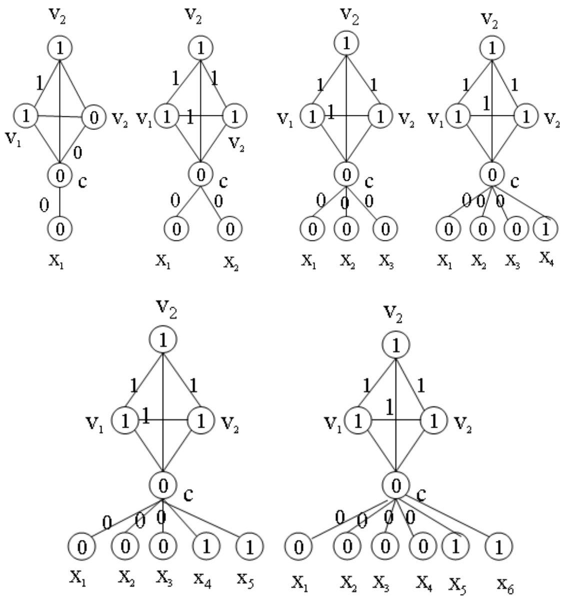


Figure 8.

Theorem 3.3. The windmill graph K(2 ^{k}, 5) is balanced if and only if k = 1,2,3,4,5,6,…...

Theorem 3.4. The windmill graph K(2 ^{k}, 6) is balanced if and only if k = 1,2,3,4,5,6,…...

Theorem 3.5. The windmill graph K(2 ^{k}, 3^2) is balanced if and only if k = 1,2,3,4.

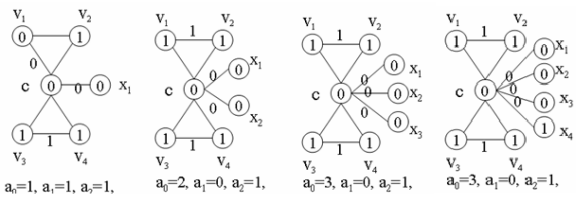


Figure 11.

Theorem 3.6. The windmill graph K(2 ^{k}, 3^3) is not balanced for all k$\geq$1.

Theorem 3.7. For n $\geq$ 3, the windmill graph K(n ^{k}, n+1) is balanced if and only if k= 1 and 2.

Proof. We show the result for n =3 and 4

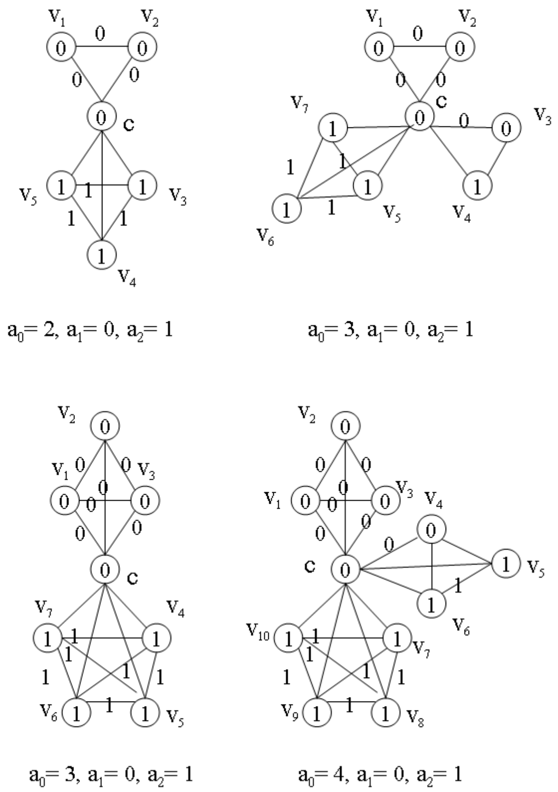
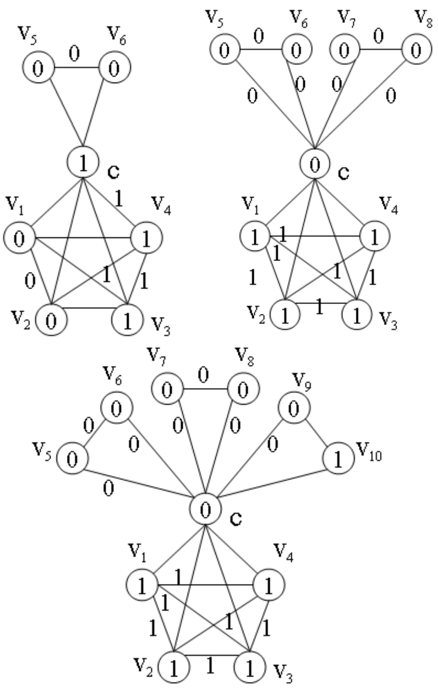


Figure 12.

Theorem 3.8. The windmill graph K(3 ^{k}, 5) is balanced if and only if k$\leq$ 5.

Proof.



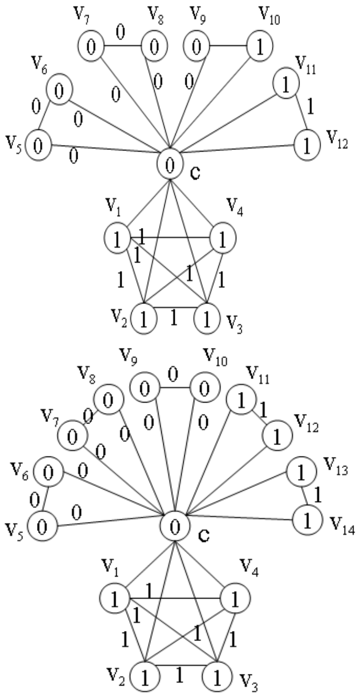


Figure 13.

Theorem 3.9. The windmill graph $K(3^k, 6)$ is balanced if and only if $k = 1$.

Proof

4. Balanced Windmill graphs with three different types of components.

Theorem 4.1. The windmill graph $K(2^k, 3, 4)$ is balanced if and only if k = 1,2,3,4,5,6,7.

Proof.

Theorem 4.2. The windmill graph $K(2^k, 3^2, 4)$ is balanced if and only if $k \leq 6$.

Proof

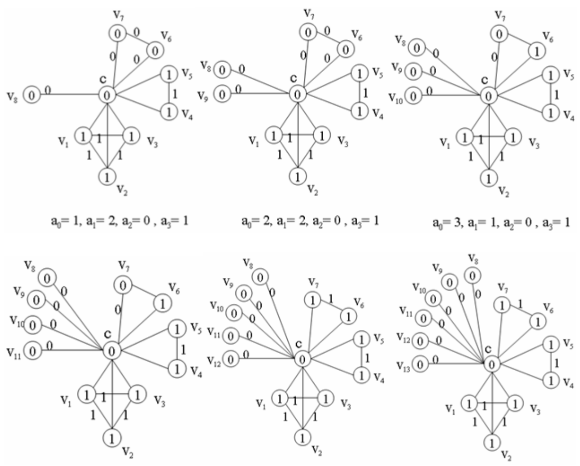


Figure 16.

Theorem 4.3. The windmill graph $K(2^k, 3, 5)$ is balanced if and only if $k \leq 12$.

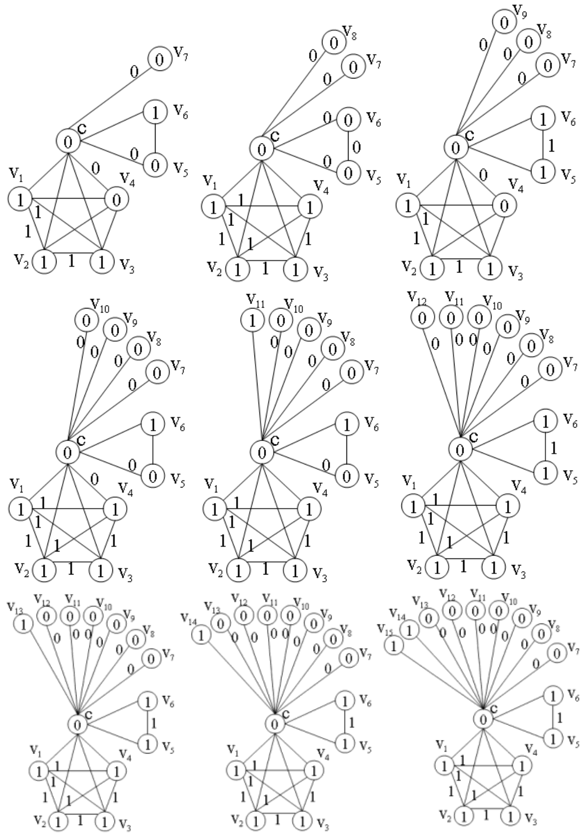
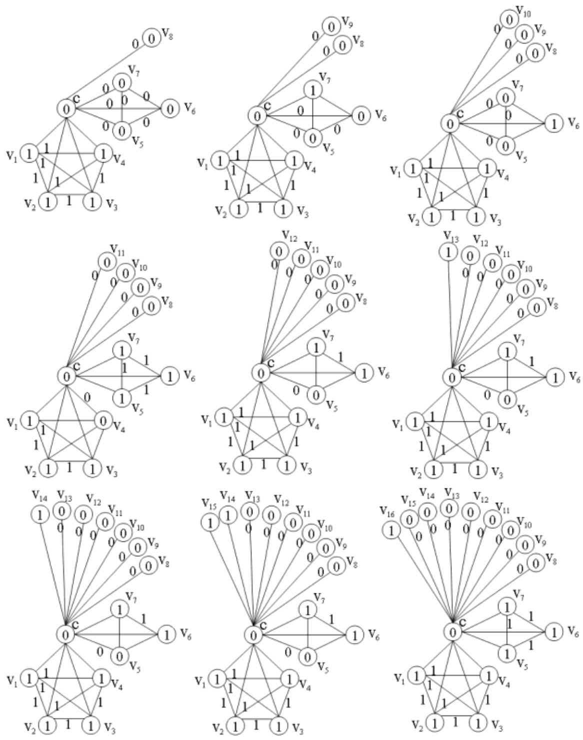


Figure 17.

Theorem 4.4. The windmill graph $K(2^k, 4, 5)$ is balanced if and only if $k \leq 15$.



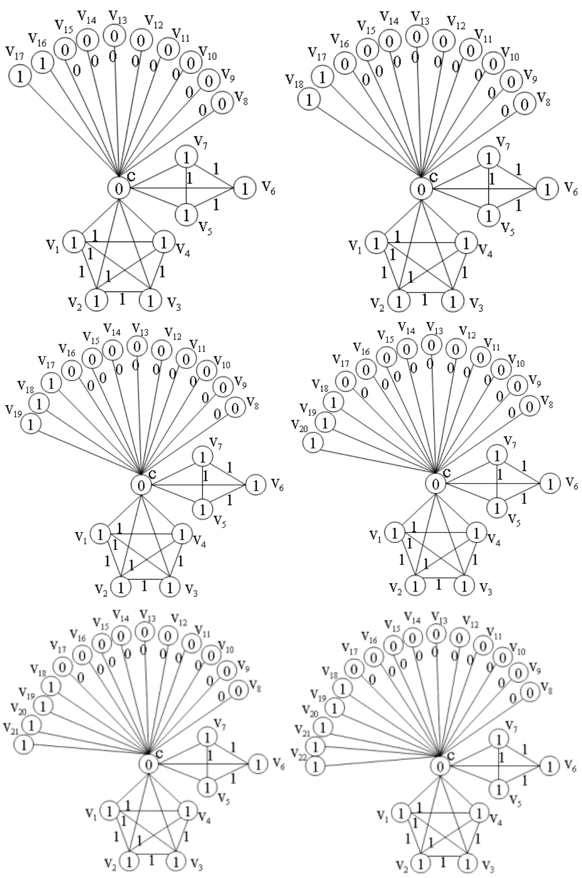


Figure 17.

Theorem 4.5. The windmill graph K(2^k, 4,6) is balanced if and only if $k \leq 15$.

References

1. M. Benson and Sin-Min Lee, On cordialness of regular windmill graphs, *Congr. Numer.*, **68**, (1989), 49-58.
2. I. Cahit, Cordial graphs : a weaker version of graceful and harmonious graphs, *Ars Combin.*, **23** (1987) 201-207.
3. I. Cahit, On cordial and 3-equitable graphs**,** *Utilitas Mathematica*, **37,** (1990),189-198.
4. I. Cahit, Recent results and open problems on cordial graphs, in *Contemporary Methods in Graph Theory*, 209-230, Bibligraphisches Inst., Mannhiem, **1990,**
5. N. Cairnie and K.Edwards, The computational complexity of cordial and equitable labelings, *Discrete Math.* **216** (2000),29-34.
6. G. Chartrand, Sin-Min Lee and Ping Zhang, Uniformly cordial graphs, *Discrete Math*. 306(2006),726-737.
7. A. Elumalai , On graceful, cordial and elegant labelings of cycle related and other graphs, Ph. D. dissertation of Anna University, 2004, Chennai, India.
8. J.A. Gallian, A dynamic survey of graph labeling, *The Electronic J. of Combin*. (2007), # DS6, 1-180.
9. Y.S. Ho , S.M. Lee, H.K. Ng and Y. H. Wen, On Balancedness of Some Families of Trees, manuscript.
10.Y.S. Ho, Sin-Min M. Lee and S.S. Shee, Cordial labellings of the Cartesian product and composition of graphs, *Ars Combinatoria* **29**(1990), 169-180.
11. Y.S. Ho, Sin- Min Lee and S.S. Shee, Cordial labellings of unicyclic graphs and generalized Petersen graphs. *Congressus Numerantium*, **68**(1989), 109-122.
12. R. Y Kim, Sin-Min Lee and H.K. Ng, On balancedness of some families of graphs, manuscript.
13. M. C. Kong, Sin-Min Lee, Eric Seah, and Alfred S. Tang, A Complete Characterization of Balanced Graphs, manuscript.
14.W.W. Kircherr, Algebraic approaches to cordial labeling .*Graph Theory, Combinatorics, Algorithms, and Applications*, Y. Alavi, et. al., editors, SIAM, (**1991**),294-299.
15. W.W. Kircherr, On cordiality of certain specified graphs, Ars Combinatoria 31 (1991),127-138.
16. W.W. Kircherr, NEPS operations on cordial graphs. *Discrete Math*., **115,** (1993), 201-209.
17. S. Kuo, G.J. Chang, Y.H.H. Kwong, Cordial labeling of mK_n, *Discrete Math*.,**169**,(1997) 121-131.
18. Alexander Nien-Tsu Lee, Sin-Min Lee and H.K. Ng, On balance index sets of graphs, manuscript.
19.Sin-Min Lee and A. Liu, A construction of cordial graphs from smaller cordial graphs, *Ars Combin*., **32** (1991) 209-214.
20.Sin-Min Lee, A. Liu and S.K. Tan, On balanced graphs, *Congressus Numerantium* 87 (1992), 59-64.
21.Y.H. Lee, H.M. Lee, G.J. Chang, Cordial labelings of graphs, *Chinese J. Math*., **20**, (1992), 3, 263-273
22. E. Seah, On the construction of cordial graphs, *Ars. Combin*. **31**,(1991),249-254.
23.M.A. Seoud and A.E.I. Abdel Maqsoud, On cordial and balanced labelings of graphs, *J. Egyptian Math. Soc*., 7 (1999) 127-135
24. S. C. Shee, The cordiality of the path-union of n copies of a graph, *Discrete Math.*, **151,** (1996), 1-3, 221-229.
25. S.C. Shee and Y.S. Ho, The cordiality of one-point union of n-copies of a graph, *Discrete Math*., **117** (1993) 225-243

