The Solution of Nonlinear Volterra Integro-Differential Equations of Second Kind by Combine Sumudu Transform and Adomain Decomposition Method

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Abstract **in this paper, we propose a combination form of the sumudu transform and Adomain decomposition method to solve nonlinear Volterra integro – differential equation of the second kind. The result reveals that the proposed method is very efficient, simple and can be applied to other applications.**
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Keywords: Volterra integro - differential equation, sumudu transform Adomain decomposition method.

I. INTRODUCTION

This It is will know that linear and nonlinear Volterra integro – differential equation arise in many scientific fields such as the population dynamic, neutron diffusion and semi conductor devices.
The Volterra integro – differential equation appears in the form

$$u^{(n)}(x) = f(x) + \lambda\int_0^x K(x,t)u(t)dt \ , \qquad (1)$$

It is our coal in this paper study the nonlinear Volterra integro – differential equations of the second kind given by

$$u^{(n)}(x) = f(x) + \int_0^x K(x,t)F(u(t))dt \ , \qquad (2)$$

Where $u^n(x)$are the nth derivative of $u(x)$, and the initial condition $u(0), u'(0), ..., u^{n-1}(0)$ are prescribed. The kernel $K(x,t)$ and the function $f(x)$ are given real- valued functions, and $F(u(t))$ is a nonlinear function of $u(x)$.
Several techniques such that variation iteration method, series solution method, and combined Laplace transform – Adomain decomposition method see [5 - 16] have been used for solving these problems. The advantage of these methods is its capability of combining the two powerful methods for obtaining exact solutions for nonlinear equations.

II. SUMUDU TRANSFORM

In early 90's, Watugala see [4] introduced a new integral transform , named the sumudu transform and applied it to the solution of ordinary differential equations in control engineering problems. The sumudu transform is defined over the set of function

$$A = \left\{ f(t) : \exists M, \tau_1, \tau_2 > 0, |f(t)| < M e^{\frac{|t|}{\tau_j}}, if\ t \in (-1)^j \times [0,\infty) \right\}$$

By the following formula

$$\bar{f}(u) = S[f(t)] = \int_0^\infty f(ut)e^{-t}\,dt \ , u \in (-\tau_1, \tau_2) \qquad (3)$$

For more details see [1- 3].

III. COMBINE SUMUDU TRANSFORM AND ADOMAIN DECOMPOSITION METHOD

To illustrate the basic idea of this method, we consider the kernel $K(x,t)$ of equation (2) as difference kernel that depends on the difference $(x-t)$.

The nonlinear Volterra integro- differential equation (2) can be expressed as

$$u^{(n)}(x) = f(x) + \int_0^x K(x-t)\,F(u(t))dt \qquad (4)$$

Consider two functions $f_1(x)$ and $f_2(x)$ that possess the conditions.

Let sumudu transform for the functions $f_1(x)$ and $f_2(x)$ given by

$$S[f_1(x)] = F_1(u) \ , \ S[f_2(x)] = F_2(u) \qquad (5)$$

The sumudu convolution product of these two functions is defined by

$$S[(f_1 * f_2)(x)] = S\left[\int_0^x f_1(x-t)\, f_2(x)\,dt\right] = u\,F_1(u)F_2(u) \qquad (6)$$

To solve the nonlinear Volterra integro- differential equation by using sumudu transform, it is essential to use the sumudu transform of the derivatives of $u(x)$ are defined by

$$S\left[u^{(n)}(x)\right] = \frac{S[u(x)]}{u^n} - \frac{u(0)}{u^n} - \frac{u'(0)}{u^{n-1}} - \dots - \frac{u^{n-1}(0)}{u} \qquad (7)$$

This simply gives

$$\begin{aligned} S[u'(x)] &= \frac{S[u(x)]}{u} - \frac{u(0)}{u} = u^{-1}U(u) - u^{-1}u(0), \\ S[u''(x)] &= u^{-2}U(u) - u^{-2}u(0) - u^{-1}u'(0), \\ S[u'''(x)] &= u^{-3}U(u) - u^{-3}u(0) - u^{-2}u'(0) - u^{-1}u''(0), \end{aligned} \qquad (8)$$

And so on for derivatives of higher order, where $U(u) = S[u(x)]$.

Applying Sumudu transform to both sides of Eq. (2) to get

$$\begin{aligned} &u^{-n}\,S[u(x)] - u^{-n}u(0) - u^{1-n}\,u'(0) - \dots - u^{-1}\,u^{n-1}(0) \\ &= S[f(x)] + u\,S[K(x-t)]S[F(u(x))] \end{aligned} \qquad (9)$$

Or equivalently

$$\begin{aligned} S[u(x)] &= u(0) + u\,u'(0) + \dots + u^{n-1}\,u^{n-1}(0) + \\ &+ u^n\,S[f(x)] + u^{n+1}\,S[K(x-t)]S[F(u(x))] \end{aligned} \qquad (10)$$

Taking the inverse sumudu transform to both sides of Eq. (10) to get

$$\begin{aligned} u(x) &= u(0) + x\,u'(0) + \dots + \frac{x^{n-1}}{(n-1)!}\,u^{n-1}(0) + \\ &S^{-1}\left[u^n\,S[f(x)]\right] + S^{-1}\left[u^{n+1}\,S[K(x-t)]S[F(u(x))]\right] \end{aligned} \qquad (11)$$

Now, we apply the Adomain decomposition method

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \qquad (12)$$

And the nonlinear terms can be decomposed as

$$F(u(x)) = \sum_{n=0}^{\infty} A_n(x) \qquad (13)$$

For some Adomain polynomials $A_n(U)$ that are given by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\left(\sum_{n=0}^{\infty}\lambda^n U_n\right)\right]_{\lambda=0}, \quad n = 0,1,2,\dots \qquad (14)$$

Substituting Eq. (12) and Eq. (13) into Eq. (11) leads to

$$\begin{aligned} \sum_{n=0}^{\infty} u_n(x) &= u(0) + x\,u'(0) + \dots + \frac{x^{n-1}}{(n-1)!}\,u^{n-1}(0) + \\ &S^{-1}\left[u^n\,S[f(x)]\right] + S^{-1}\left[u^{n+1}\,S[K(x-t)]S\left[\sum_{n=0}^{\infty} A_n(x)\right]\right] \end{aligned} \qquad (15)$$

So that the recursive relation is given by

$$\begin{aligned} u_0(x) &= u(0) + x\,u'(0) + \dots + \frac{x^{n-1}}{(n-1)!} + S^{-1}\left[u^n\,S[f(x)]\right], \\ u_{k+1}(x) &= S^{-1}\left[u^{n+1}\,S[K(x-t)]S[A_k]\right], \quad k \ge 0\,. \end{aligned} \qquad (16)$$

IV. Numerical Applications

The combined sumudu transform – Adomain decomposition method for solving nonlinear Volterra integro- differential equations of the second kind will be illustrated by studding the following examples.

Example **1**: consider the initial value problem

$$\begin{aligned} u'(x) &= \frac{9}{4} - \frac{5}{2}x - \frac{1}{2}x^2 - 3e^{-x} - \frac{1}{4}e^{-2x} + \\ &+ \int_0^x (x-t)\,u^2(t)\,dt, \quad u(0) = 2 \end{aligned} \qquad (17)$$

Notice that the kernel $K(x-t) = (x-t)$. Taking sumudu transform of both sides of Eq. (17) gives

$$\begin{aligned} S[u'(x)] &= S\left[\frac{9}{4} - \frac{5}{2}x - \frac{1}{2}x^2 - 3e^{-x} - \frac{1}{4}e^{-2x}\right] + \\ &+ S\left[(x-t) * u^2(x)\right]. \end{aligned} \qquad (18)$$

So that

$$\begin{aligned} u^{-1}U(u) - u^{-1}u(0) &= \frac{9}{4} - \frac{5}{2}u - u^2 - \frac{3}{1+u} - \frac{1}{4(1+2u)} + \\ &+ u^2 S\left[u^2(x)\right], \end{aligned} \qquad (19)$$

Or equivalently

$$\begin{aligned} U(u) &= 2 + \frac{9}{4}u - \frac{5}{2}u^2 - u^3 - \frac{3u}{1+u} - \frac{u}{4(1+2u)} + \\ &+ u^3 S\left[u^2(x)\right], \end{aligned} \qquad (20)$$

Applying the inverse sumudu transform to both sides of Eq. (20) gives

$$\begin{aligned} u(x) &= 2 + \frac{9}{4}x - \frac{5}{4}x^2 - \frac{x^3}{3!} - 3 + 3e^{-x} - \frac{1}{8} + \frac{1}{8}e^{-2x} + \\ &+ S^{-1}\left[u^3\,S\left[u^2(x)\right]\right], \end{aligned} \qquad (21)$$

Or equivalently

$$u(x)=2-x+\frac{1}{2}x^2-\frac{5}{3!}x^3+\frac{5}{4!}x^4-\frac{7}{5!}x^5+ +.....+S^{-1}\left[u^3 S\left[u^2(x)\right]\right] \quad (22)$$

Substituting the series assumption for $u(x)$ and the Abomain polynomials for $u^2(x)$ as given above in Eq. (12) and Eq. (13) respectively, and using the recursive relation to obtain

$$u_0(x)=2-x+\frac{1}{2}x^2-\frac{5}{3!}x^3+\frac{5}{4!}x^4-\frac{7}{5!}x^5+..... ,$$
$$u_{k+1}(x)=S^{-1}\left[u^3 S[A_K]\right],\ k\geq 0 . \quad (23)$$

Recall that the Adomain polynomials for $F(u(x))=u^2(x)$ are given by

$$A_0=u_0^{\ 2},$$
$$A_1=2u_0u_1,$$
$$A_2=2u_0u_2+u_1^{\ 2}, \quad (24)$$
$$A_3=2u_0u_3+2u_1u_2 .$$

Substituting these polynomials into the recursive relation to find

$$u_0(x)=2-x+\frac{1}{2}x^2-\frac{5}{3!}x^3+\frac{5}{4!}x^4-\frac{7}{5!}x^5+....$$
$$u_1(x)=\frac{2}{3}x^3-\frac{1}{6}x^4+\frac{1}{20}x^5+.... \quad (25)$$

Using (12), to find the series solution of eq. (17), in the form

$$u(x)=2-x+\frac{1}{2}x^2-\frac{1}{3!}x^3+\frac{1}{4!}x^4-\frac{1}{5!}x^5+.... \quad (26)$$

That converges to the exact solution

$$u(x)=1+e^{-x} . \quad (27)$$

Example2: consider the following integro-differential equation

$$u''(x)=-1-\frac{1}{3}(\sin x+\sin 2x)+2\cos x+ +\int_0^x \sin(x-t)u^2(t)dt, u(0)=-1, u'(0)=1 \quad (28)$$

Taking Sumudu transform of (28), to find

$$S[u''(x)]=S\left[-1-\frac{1}{3}(\sin x+\sin 2x)+2\cos x\right]+ +S\left[\sin(x-t)*u^2(x)\right], \quad (29)$$

So that

$$u^{-2}U(u)-u^{-2}u(0)-u^{-1}u'(0)= -1-\frac{u}{3(1+u^2)}-\frac{2u}{3(1+4u^2)}+\frac{2}{1+u^2}+\frac{u^2}{1+u^2}S\left[u^2(x)\right], \quad (30)$$

Or equivalently

$$U(u)=-1+u-u^2-\frac{u^3}{3(1+u^2)}-\frac{2u^3}{3(1+4u^2)}+ +\frac{2u^2}{1+u^2}+\frac{u^4}{1+u^2}S\left[u^2(x)\right], \quad (31)$$

Applying inverse Sumudu to both sides of Eq. (31) gives

$$u(x)=-1+x+\frac{x^2}{2!}-\frac{1}{3!}x^3-\frac{1}{12}x^4+\frac{1}{40}x^5+ +\frac{1}{360}x^6-\frac{11}{5040}x^7+.......+S^{-1}\left[\frac{u^4}{1+u^2}S\left[u^2(x)\right]\right] \quad (32)$$

Proceeding as before we find

$$u_0(x)=-1+x+\frac{x^2}{2!}-\frac{1}{3!}x^3-\frac{1}{12}x^4+\frac{1}{40}x^5+\frac{1}{360}x^6-...$$
$$u_1(x)=\frac{1}{4!}x^4-\frac{1}{60}x^5-\frac{1}{720}x^6+\frac{1}{504}x^7+...$$

(33)

Using (12), to find the series solution of eq. (28), in the form

$$u(x)=\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+......\right)-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+........\right), \quad (34)$$

That converges to the exact solution

$$u(x)=\sin x-\cos x . \quad (35)$$

V. CONCLUSIONS

In the present paper, we have combined form of Sumudu transform with Adomain decomposition method is effectively used to solve nonlinear Volterra integro- differential equations of the second kind.

From the examples considered here, it can be easily seen that this method obtains results as accurate as possible.

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