BER Performance of ZF AND MMSE Equalizers for MIMO Systems

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***Abstract*— This paper presents the comparison of probability of bit error rate between the zero forcing (ZF) and minimum mean squared error (MMSE) equalizers applied to wireless multi-input multi-output (MIMO) systems. Contrary to the common perception that ZF and MMSE are asymptotically equivalent at high SNR, it shows that the output SNR of the MMSE equalizer (conditioned on the channel realization) is $\rho_{mmse} = \rho_{Zf} + \eta_{SNR}$, where ρ_{Zf} is the output SNR of the ZF equalizer, and that the gap η_{SNR} is statistically independent of ρ_{Zf} and is a non-decreasing function of input SNR. Furthermore, as SNR → ∞, η_{SNR} converges with probability one to a scaled f random variable. It is also shown that at the output of the MMSE equalizer, the Interference-to-noise ratio (INR) is tightly upper bounded by $\frac{\eta_{SNR}}{\rho_{Zf}}$.**

Keywords— **Zero forcing, minimum mean squared error, MIMO, BER,**

I. INTRODUCTION

Consider the complex baseband model for wireless multi-input multi-output (MIMO) channel with N_t transmit antennas and N_r receiver antennas Y=Hx+z, where Y is the received signal and H is a Rayleigh fading channel with independent, identically distributed (i.i.d) [4,5], circularly symmetric standard complex Gaussian entries, denoted as $h_{ij} \sim N_r(0, 1)$ for $1 \leq i \leq N_t$; $1 \leq j \leq$ Nr. [1]. We also assume that the *nr* data sub streams have uniform power, i.e.,$x \in \sigma^{N_r \times 1}$has covariance matrix $E[xx^*] = \sigma_x^2 I_{N_r}$, where E[.] stands for the expected value, $(.)^*$ is the conjugate transpose, and I_{N_r} is an $N_t \times N_r$ identity matrix. The white Gaussian noise $z \sim N_r(0,\sigma_z^2 I_{N_r})$. is also circularly symmetric. The input signal-to-noise ratio (SNR) is defined as SNR = $\frac{\sigma_x^2}{\sigma_z^2}$.This paper presents the comparison of probability of bit error rate between the zero forcing (ZF) and minimum mean squared error (MMSE) equalizers applied to to the channel given in (1). we present an in-depth analysis of the performance of the zero forcing (ZF) and minimum mean squared error (MMSE) equalizers applied to the channel Y=Hx+z. The linear ZF and MMSE equalizers are classic functional blocks and are ubiquitous in digital communications [1]. Despite their fundamental importance, however, the existing performance analyses of the ZF and MMSE equalizers are far from complete. For instance, it is commonly understood that ZF is a limiting form of MMSE as SNR → ∞ . But when the ZF and MMSE are applied to the MIMO fading channel given in (1), one may observe through simulations that the error probabilities of MMSE and ZF do not coincide even as SNR → ∞.

The major findings of this paper are given below.

A common perception about ZF and MMSE is that ZF is the limiting form of MMSE as SNR → ∞. Therefore, it is presumed that the two equalizers would share the same output SNRs, and consequently, the same uncoded error in the high SNR regime. However, The output SNRs of the N data substreams using MMSE and ZF are related by

$$\rho_{mmse,\,n} = \rho_{Zf,\,n} + \eta_{SNR,\,n} \;,\; 1 \leq n \leq N \qquad (1)$$

where $\rho_{Zf,\,n}$ and $\rho_{mmse,\,n}$ are statistically independent and η_{SNR} is a non-decreasing function of SNR. Moreover, $\eta_{SNR,\,n} \rightarrow \eta_{\infty,n}$ as SNR→ ∞, where $\frac{N_r - N_r + 2}{N_t - 1}\eta_{\infty,\,n} \sim f_{2(N_t-1),(N_r-N_r+2)}$ is of f-distribution. Further, the Interference-to-noise ratio (INR) of the nth substream at the output of MMSE (denoted as inr_n) , is approximately upper bounded as $inr_n = \frac{\eta_{SNR,n}}{\rho_{Zf,}}$.With the approximate upper bound being asymptotically tight for high SNR.Since $inr_n=\frac{\eta_{SNR,n}}{\rho_{Zf,n}}$ is inversely proportional to the input SNR, (5) implies that the higher the input SNR, the smaller the leakage from the interfering substreams.

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II. BASICS OF ZF AND MMSE EQUALIZERS

Consider the MIMO channel model given in (1) where the *N* data sub streams are mixed by the channel matrix. The ZF and MMSE equalizers can be applied to decouple the *N* sub streams. The ZF and MMSE equalization matrices are

$$W_{Zf}=(H^*H)^{-1}H^* \text{ and } W_{mmse}=(H^*H+\frac{1}{SNR}I)^{-1}H^* \qquad (2)$$

Let multiplying the received signal vector Y by W_{Zf} and W_{mmse} , we obtain *N* decoupled substreams with output SNRs

$$\rho_{Zf,n} = \frac{SNR}{[(H^*H)^{-1}]_{nn}}, \quad 1 \leq n \leq N_r$$

$$\rho_{mmse,n} = \frac{SNR}{[\left(H^*H+\frac{1}{SNR}I\right)^{-1}]_{nn}} - 1, \quad 1 \leq n \leq N_r \qquad (3)$$

Here $[.]_{nn}$ denotes the n th diagonal element. Denote h_n the nth column of H and H_n the submatrix obtained by striking h_n out of H. Hence

$[(H^{*}H)]_{nn} = h_n{}^{*}h_n - h_n{}^{*}H_n(H_n{}^{*}H_n)^{-1}H_n{}^{*}h_n$

That

$$\rho_{Zf,n} = [(h_n{}^{*}h_n - h_n{}^{*}H_n(H_n{}^{*}H_n)^{-1}H_n{}^{*}h_n)^{-1}]SNR, \quad (4)$$

where $I - H_n(H_n{}^{*}H_n)^{-1}H_n{}^{*}$ stands for the orthogonal projection onto the null space of $H_n{}^{*}$. In the case of i.i.d. Rayleigh fading, $h_n{}^{*}(I - H_n(H_n{}^{*}H_n)^{-1}H_n{}^{*})\,h_n \sim \chi^2_{2(N_t-N_r+1)}$, with distribution

$$f_{h_n{}^{*}(I-H_n(H_n{}^{*}H_n)^{-1}H_n{}^{*})h_n}(\chi) = \frac{1}{(N_t-N_r)!}\chi^{(N_t-N_r)}e^{-\chi}, \quad \chi \geq 0 \quad (5)$$

Similarly, we have an alternative expression for $\rho_{mmse,n}$ is

$$\rho_{mmse,n} = [(h_n{}^{*}h_n - h_n{}^{*}H_n\left(H_n{}^{*}H_n + \frac{1}{SNR}I\right)^{-1}H_n{}^{*}h_n)^{-1}]SNR, \quad 1 \leq n \leq N_r. \quad (6)$$

III. Analysis of the Output SNR of MMSE

Since the elements of the channel matrix H are i.i.d., the output SNRs of the N substreams are of identical (but not independent) marginal distributions. Hence, to study the distribution of the output SNRs of the N substreams, we only need to focus on one, say the nth substream. Starting with analyzing the gap between the output SNRs of ZF and MMSE. The diference between $\rho_{mmse,n}$ and $\rho_{Zf,n}$ denoted by η_{SNR} is

$$\eta_{SNR,n} = \rho_{mmse,n} - \rho_{Zf,n} = SNR\, h_n{}^{*}H_n[(H^{*}H)^{-1} - \left(H_n{}^{*}H_n + \frac{1}{SNR}I\right)^{-1}]H_n{}^{*}h_n \quad (7)$$

Intuitively, $\eta_{\infty,n}$ represents the power of the signal component \hiding" in the range space of H_n that is recovered by the MMSE equalizer. In contrast, the ZF equalizer nulls out that signal component. For any full rank channel matrix, $\frac{\eta_{SNR,n}}{\rho_{Zf,n}} \to 0$ as SNR→ ∞.Therefore, the interference from the other data substreams is negligible compared to the channel noise as SNR →∞. Consequently, for any full rank channel realization, the ratio of the output SNR gains (in dB) of the MMSE to ZF equalizers goes to unity or

$$10\log_{10}\left(\frac{\rho_{mmse,n}}{\rho_{Zf,n}}\right) = 10\log_{10}\left(1 + \frac{\eta_{SNR,n}}{\rho_{Zf,n}}\right) \to 0 \text{ as SNR} \to \infty$$

In spite of the diminishing relative output SNR gain, the MMSE is shown to have remarkable SNR gain over ZF even as SNR→∞ owing to the fact that the limit of their difference is an f- random variable. In recovering the signal χ_n in in the range space of H_n, the MMSE equalizer admits some leakage from the other interfering data sub-streams. It is shown that the leakage diminishes as input power increases because the INR at the output of the MMSE equalizer is in fact inversely proportional to the input SNR.

IV. UNCODED ERROR PROBABILITY ANALYSIS

The error probability of n[th] sub-stream obtained by ZF equalizer is

$$P_{b,Zf} = \left[\frac{1}{2}\left(1 - \sqrt{\frac{SNR}{1+SNR}}\right)\right]^{N_t-N_r+1} \times \sum_{n=0}^{N_t-N_r+1}\binom{N_t - N_r + 1}{n}\left(\frac{1+\sqrt{\frac{SNR}{1+SNR}}}{2}\right) \quad (8)$$

And the error probability of MMSE in terms of $P_{b,Zf}$ is

$$P_{b,mmse} = E\left[Q\left(\sqrt{2\left(\rho_{Zf,n} + \eta_{\infty,n}\right)}\right)\right] = E\,[e^{-\eta_{\infty,n}}]P_{b,Zf} \quad (9)$$

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V. RESULTS AND DISCUSSION

Simulation of MIMO for Computing Minimum Mean Square Error (MMSE):

The simulation is being done by using the MATLAB and the step by step procedure to find the results of bit error rate with the prescribed concepts is being described. Fig.1 shows the bit error rate with MMSE equalizer, BPSK modulation and 2×2 MIMO system. It is clearly seen from the simulation result that, in increase SNR the bit error get decrease.

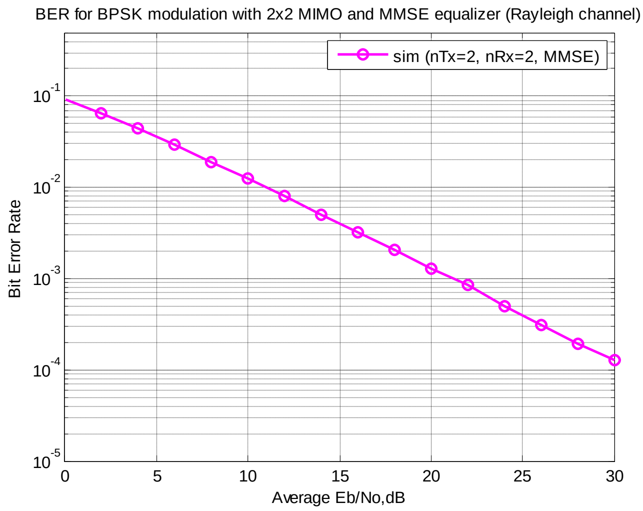

Figure 1: BER with MIMO and MMSE

Simulation of MIMO for Computing Zero Forcing Equalizer:

Fig.2 shows the bit error rate with MMSE equalizer, BPSK modulation and 2×2 MIMO system. It is clearly seen from the simulation result that, in increase SNR the bit error get decrease.

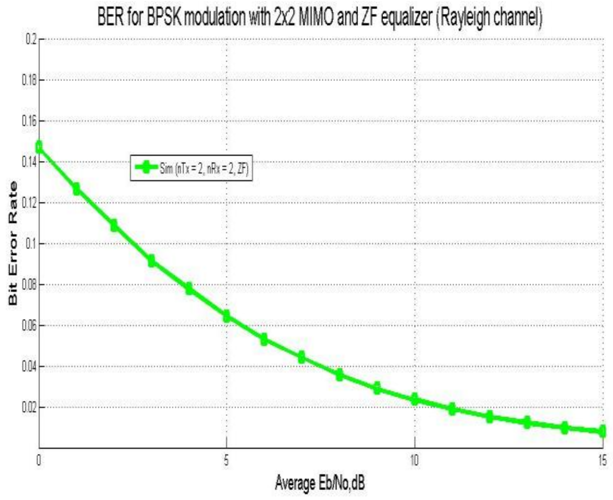

Figure 2: BER with MIMO and ZF

Simulation of Comparison between ZF and MMSE for MIMO

Fig.3 shows the comparison between the bit error rate of ZF and MMSE equalizer, BPSK modulation and 2×2 MIMO system. It is clearly seen from the simulation result that MMSE gives the better performance than ZF.

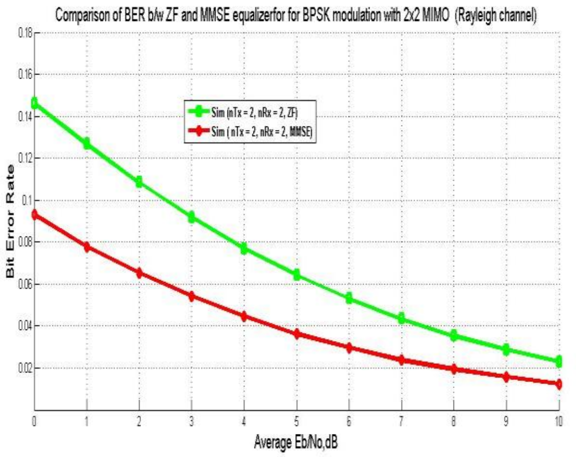

Figure 3: BER comparison with MIMO between MMSE and ZF

VI. CONCLUSION

This paper analyzed the performances of the zero forcing (ZF) and minimum mean squared error (MMSE) equalizers applied to a 2×2 wireless multi-input multi-output (MIMO) systems, in terms of output SNR, uncoded error. It has been shown that there is a gap between the output SNRs of ZF and MMSE equalizers, which converges with probability one to a random variable of scaled *F*-distribution as input SNR goes to infinity.

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