Analysis for the Parameters of various Queueing Models

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***Abstract*– The queueing theory has been analysed by several researchers in different angles. There are different queueing models and differ individually in the nature of the models. In this paper the variations of Markovian and non Markovian queues are discussed based on single service as well as bulk service. This studied models are governed by either single server or multi server. The effectiveness of queueing models are compared with other models based on some criteria. The steady state probabilities and expected queue length are used to compare the models through numerical results and related curves.**

***Keywords*– Service channels, Steady-state probabilities, Markovian queue , multi-server.**

I. INTRODUCTION

Various types of queueing systems have been studied by several authors in different angles during the past 10 decades. Some contributors have discussed the queues of the types, namely Markovian, Erlangian, Hyper-Poisson, Deterministic and so on. A queue can be called as one of the above terms on the basis of the pattern of arrival or departure or both. A queue is maintained by either single or multi server. The customers arrive either singly or bulk and they are also served either in singly or bulk. The standard bulk rules are (i) usual (ii) general (iii) accessible and (iv) Non accessible.

Different methods to solve the queueing problems have been introduced in the literature.

In this paper, consider the queueing systems in which the service is rendered one by one or bulk and are governed by single or multi server. The above said models are taken into account and discuss the variations between the models on the light of the steady-state probabilities and expected queue lengths.

II. SINGLE SERVER WITH SINGLE SERVICE

In this section, consider single server queues in which service is happened one by one and the models which are to be discussed here are M/M/1,$E_2/E_2/1$ and $HE_n/M/1$.

Comparing these models by using the steady-state probabilities and expected queue lengths.

A. M/M/1 queueing model

Gross and Harris (1974) have studied a single server queueing system in which the arrival process is poisson with mean arrival rate λ and the service distribution is exponential with mean service rate μ.

Let $P_n(t)$ be the probability that there are 'n' customers in the system at time t.

The resultant differential – difference equations for M/M/1 model are

$$P_n^1(t) = -(\lambda+\mu)P_n(t) + \mu P_{n+1}(t) + \lambda P_{n-1}(t), \quad n \geq 1 \qquad \text{\_\_\_\_\_\_\_\_\_} \quad (1)$$

$$P_0^1(t) = -\lambda P_0(t) + \mu P_1(t) \qquad \text{\_\_\_\_\_\_\_\_\_} \quad (2)$$

By taking the limit as $t \to \infty$

Then $P_n(t) \to P_n$ and $P_n^1(t) \to 0$.

Using the above conditions in equations (1) and (2), and get the steady – state probabilities as

$$P_n = \rho^n(1-\rho), \qquad \rho = \frac{\lambda}{\mu}, n \geq 0 \qquad \text{\_\_\_\_\_\_\_\_\_} \quad (3)$$

After using some algebra in (3), the expected number of customers in the system and that in the queue are given respectively as follows.

$$L = \rho/(1-\rho) \qquad \text{\_\_\_\_\_\_\_\_\_} \quad (4)$$

and

$$Lq = \frac{\rho^2}{1-\rho} \qquad \text{\_\_\_\_\_\_\_\_\_} \quad (5)$$

$E_2/E_2/1$ queueing model

Morse (1958) has discussed the more complicated case where arrivals are I-Erlang and the service channels are k-Erlang. Here I and k are the maximum number of arrival stages and service phases respectively. The arrival stages are numbered in order of the customer's progress whereas the service phases are numbered in reverse order.

If I=k=1, then the model reduced to M/M/1 model.

When l=k=2, the above model becomes $E_2/E_2/1$ and its steady – state probabilities are given by

$$P_n = C_1[-v_1^{2n-2}(1+V_1)^2 + (1+\rho+V_1)\ \rho^{2n-2}(1+\rho)^2] \qquad (6)$$

Where, $v_1 = \frac{1}{2}(1+\rho) - \frac{1}{2}\sqrt{1+6\rho+\rho^2}$

and $V_2 = \rho$

Are the roots of a secular equation

$$P_0 = \left(\frac{C_1}{\rho}\right)[3\rho + \rho^2 + (1+\rho)\ v_1] \qquad (7)$$

$$C_1 = \left[3 + \rho + \frac{1+\rho}{\rho} V_1 + (1+\rho+V_1)\frac{1+\rho}{1-\rho} - \frac{1+V_1}{1-V_1}\right]^{-1}$$

By applying little algebra in (6), the expected number of customers in the system is given by

$$L = C_1\left[-\left(\frac{1+V_1}{1-v_1^2}\right)^2 + (1+\rho+V_1)\left(\frac{1+\rho}{1-\rho^2}\right)^2\right] \qquad (8)$$

HE$_n$/M/1 queueing model

Suppose that the arrival channel consists of n independent branches. Only one customer can enter any branch at a time and if there is a customer present in any one of the n branches, no other customer can enter any other branch. Further assume that there is a reservoir of infinite capacity that emits a customer as soon as the arrival channel is free. The customer in the rth branch enters the system at the rate λ_r per unit time, the queue discipline being "first come, first served". The arriving customers are served one by one according to the exponential distribution with parameter μ. For the purpose of obtaining numerical solutions, the number of branches 'n' is taken as 2.

Morse (1958) has given the steady-state probabilities, obtained from the system of equations, as follow:

$$P_n = (1-V)\rho V^{n-1}, n \geq 1 \qquad (9)$$

$$P_0 = 1 - \rho \qquad (10)$$

The expected number of customers in the system is

$$L = {}^{\rho}/_{(1-V)} \qquad (11)$$

where

$$V = \frac{1}{2} + \rho - \sqrt{\frac{1}{4} - (2\sigma-1)^2\rho(1-\rho)}$$

For different values of ρ, ($\rho = 0.10, 0.20, 0.25, 0.40, 0.50, 0.60, 0.7$), the steady-state probabilities $P_n, n = 0$, 1,2,3, are measured for the above said three models, by using the expressions (3), (6), (7), (9) and (10), and given in table : 1. The related curves are exhibited in chart: 1.

Chart:1 shows the variations of the steady-state probabilities for M/M/1 $E_2/E_2/1$ and $HE_n/M/1$. These systems are respectively noted by the symbols M,E and H in the following statements.

The curves of the chart : 1. reveal that

(i) The steady-state probabilities for Markovian and Hyper Poisson arrival queues are equal but steady-state probabilities for Erlangian queue is lessthan that of other two queues, (ie, E<M=H), when the system is empty.

(ii) The steady-state probabilities for the three models form an inequality H<M<E when there is only one customer in the system.

(iii) The steady-state probabilities form an inequality E<M<H for least values of ρ but the inequality changes in the reverse order (H<M<E) for higher values of ρ when n = 2 and 3.

The above statements imply that the steady-state probabilities for Erlangian queue is less than that of others when the system is empty as well as for least values of ρ when $n \geq 2$. Also the steady-state probabilities for Hyper-poisson arrival queue is less than that of others for larger ρ. In all cases, Markovian queue occupies the central position.

The expected number of customers in the system (L) for the above three models, by using the expressions (4), (8) and (11), corresponding to the various values of ρ are calculated and tabulated in table : 2. The resultant curves are shown in chart : 2

The curves of the chart : 2 reveal that the expected number of customers in the system increase as the values of ρ increase. The expected number of customers in the system of $E_2/E_2/1$ is less than that of M/M/1 for $\rho > 0.25$, but this situation is reversed for $\rho \leq 0.25$. The expected number of customers in the system of $HE_n/M/1$ is greater than that of other two models and also rapidly increase for $\rho \geq 0.4$. The above statements imply that any server is unable to manage the queue in $HE_n/M/1$ for higher ρ.

B. Multi-Server with Single Service

Consider a multi-server (c) queue with poisson arrival and the service time distribution is exponential. The arriving customers are served one by one.

Gross and Harris (1974) have derived the following results.

Let $P_n, n \geq 0$ be the steady-state probabilities for n customers in the system of the model M/M/c and are given by

$$P_n = \begin{cases} \frac{1}{n!}(c\rho)^n P_0, n < c \\ \frac{1}{c^{n-c}c!}(c\rho)^n P_0, n \geq c \end{cases} \qquad \text{\_\_\_\_\_\_\_\_ (12)}$$

$$P_0 = \frac{1-\rho}{1+\rho} \qquad \text{\_\_\_\_\_\_\_\_ (13)}$$

After using little algebra, the expected number of customers in the queue (Lq) are obtained as

$$Lq = \frac{\rho}{(1-\rho)^2}, \frac{(c\rho)^c}{c!} P_0 \qquad \text{\_\_\_\_\_\_\_\_ (14)}$$

The numerical values for the steady-state probabilities (P_n) for M/M/c (c = 2) are computed, by using the expressions (12) and (13), for different values of ρ and presented in table : 3. By using the steady-state probabilities of M/M/1 (table : 1) and M/M/c (table : 3), the relevant curves are drawn in chart : 3.

Chart : 3 explains that the steady – state probabilities for M/M/1 are greater than that for M/M/c when the system is empty. On the other hand, the steady-state probabilities for the single server Markovian queue are less than that for multi-server Markovian queue when $n \geq 1$.

The expected number of customers in the queue for both models M/M/1 and M/M/c, ($c = 2$) are measured, by using the expressions (5) and (14) and tabulated in table : 4. The two curves are exhibited in chart : 4.

The chart : 4 reveals that the expected number of customers in the queue for both M/M/1 and M/M/c increase as the values of ρ increase. The expected number of customers in the queue for M/M/c ($c = 2$) are less than that for M/M/1. This implies that the expected number of customers waiting in the queue decrease as the number of servers increase.

C. Single Server with Bulk Service

The single and multi-server queues with single service have been discussed in sections (2) and (3) respectively. Here, the single server queue with bulk service is considered. There are many bulk service rules in which most frequently used types are briefly given below.

(i) The customers are served in batches of not more than b. The server may wait, if he finds none in queue on completion of a service, till there is atleast one customer available for service. In short, the size of bulk is denoted as X, $1 \leq X \leq b$. Bailey (1954) has introduced this method and named as usual bulk service rule.

(ii) A service batch may of a fixed size 'b'. The server will wait till the number reaches 'b' and if there are more than 'b', he takes a batch of b, while others wait.

(iii) A batch may contain a minimum number of 'a' and a maximum of 'b' customers (Neuts (1967)). If immediately, after completion of service of a batch, server finds less than 'a' customers present, he waits till there are 'a', where upon he takes a batch of 'a' for service; if he finds more than 'a' customers present but at most 'b', he takes them all in a batch and if he finds more than 'b' customers waiting, he takes only 'b' customers for service while others wait in the queue. In short, this bulk rule is denoted a k, $a \leq k \leq b$ and named as general bulk service rule.

(iv) Bulk service may be with accessible batches (Newell (1960)). If a batch being served does not utilize its full capacity for service, it may remain accessible for customers arriving during the service time of a batch until its full capacity is attained; the total service time is not altered by inclusion of such joining customers in course of on going service.

By adopting the above bulk service rules, many researchers have discussed various gueueing models and derived many results. The steady state probabilities obtained from Markovian model, when applying usual and general bulk service rules, are given by Medhi (1984) as follows:

For $M/M^{1,b}/1$ model,

$$P_n = (1-r)r^n, n \geq 0 \qquad \text{\_\_\_\_\_\_\_\_ (15)}$$

For $M/M^b/1$ model

$$P_n = \begin{cases} \frac{1}{b}(1-r^{n+1}), 0 \leq n \leq b-1 \\ \rho\,(1-r)r^{n-b}, n \geq b \end{cases} \qquad \text{\_\_\_\_\_\_\_\_ (16)}$$

For $M/M^{a,b}/1$ model,

$$P_{1,n} = \left(\frac{1-r^b}{1-r}\right) r^{n+1} P_{0,0}\, n \geq 0 \qquad \text{\_\_\_\_\_\_\_\_ (17)}$$

$$P_{0,0}^{1} = \frac{a}{1-r} + \frac{r^{a+1} - r^{b+1}}{(1-r)^2} \qquad \text{\_\_\_\_\_\_\_\_\_\_ (18)}$$

where 'r' is the unique positive real root of the characteristic equation:

$$\mu z^{b+1} - (\lambda + \mu)z + \lambda = 0 \qquad \text{\_\_\_\_\_\_\_\_\_\_ (19)}$$

By using the expressions (15) to (18), we have to compute the steady-state probabilities for various models, after determining the positive real root of the equation (19) and tabulated in table : 5. The related curves are exhibited in chart : 5 and the curve for equation (3) is also exhibited in the same chart. The curves of the chart : 5 indicate the variations of Markovian equeues with various bulk services as well as single service.

The descriptions are:

(i) Steady-state probabilities for the Markovian queue with general bulk service are less than that for Markovian queue with both fixed and usual bulk services.

(ii) Steady-state probabilities for the Markovian queue with fixed size bulk service are less than that for Markovioan queue with usual bulk service when null customers in the system. As $n(\geq 1)$ increases, the above statement occurs in reverse order.

(iii) Steady-state probabilities for M/M/1 are greater than that of the single server Markovian queue with general bulk service but less than that of Markovian queue with fixed and usual bulk service when n = 1, 2. For higher values of $n(> 2)$, the steady-state probabilities for M/M/1 are less than that of Markovian queue with any type of bulk services.

III. MULTI-SERVER WITH BULK SERVICE

A. $M/M^{1,b}/c$ model

Consider a queueing system in which the customers arrive according to Poisson process with mean arrival rate λ and the arriving customers are served in batches of size $k,\ 1 \leq k \leq b$ per server in the system of c servers.

The service time distribution of each batch is exponential with mean rate μ. Under these assumptions, Borthakur and Gohain (1982) have derived the following steady-state probabilities.

$$P_{0,n} = \frac{(\lambda/\mu)^c}{c!} r^n P_{0,0}, \quad n \geq o \qquad \text{\_\_\_\_\_\_\_\_\_\_ (20)}$$

and

$$P_{0,0}^{\,1} = \frac{(\lambda/\mu)^c}{c!\,(1-r)} + \sum_{m=0}^{c-1} \frac{(\lambda/\mu)^m}{m!} \qquad \text{\_\_\_\_\_\_\_\_\_\_ (21)}$$

where r is the unique positive real root of the characteristic equation.

$$c\mu z^{b+1} - (\lambda + c\mu)z + \lambda = 0 \qquad \text{\_\_\_\_\_\_\_\_\_\_ (22)}$$

After obtaining the positive real root of the equation (22), the required steady-state probabilities are measured from the expressions (20) and (21) and given in table : 5 and the curve is exhibited in chart : 5.

The curves relating to $M/M^{1,b}/c$ and $M/M^{1,b}/1$ reflect that the steady-state probabilities for $M/M^{1,b}/c$ are less than that of $M/M^{1,b}/1$. The curves of chart : 5 give the fact that the multi-server Markovian queue with usual bulk service is effective than single server Markovian queue with bulk and single services.

B. $E_2/E_2^{a,b}/c$ model

A queueing system in which the arrivals are having passed through 2 stages, each with mean arrival rate 2λ. The arriving customers are served in batches of size $k, a \leq k \leq b$, per server in the system of c servers. The service time distribution of each batch is assumed to be Erlang (type 2) family with mean service rate 2 μ. Under these assumptions, Sathiamoorthi and Ganesan (1989) have formulated the differential difference equations of $E_2/E_2^{a,b}/c$ model and derived the steady-state probabilities.

$$P_{c,n} = \frac{\lambda}{\mu}\left[A(-1)^{c-1}(1+q_2)^2 q_2^{2n+c-2} - (1+q_1)^2\, q_1^{2n+c-2}\right]$$
$$\left[A(-1)^{c-1}\left\{2\left(\frac{\mu}{\lambda+\mu}\right)^2 \sum_{k=a}^{b} q_2^{2k-2} + \frac{1}{\lambda}(\lambda+\mu-\mu q_2)\right\} - \left\{2\left(\frac{\mu}{\lambda+\mu}\right)^2 \sum_{k=a}^{b} q_1^{2k-2} + \frac{1}{\lambda}(\lambda+\mu-\mu\, q_1)\right\}\right] P_{o,o}^{-1} \qquad \text{\_\_\_\_\_\_\_\_ (23)}$$

$$P_{0,0} = \frac{\mu}{\lambda}\left[A(-1)^{c-1}\left\{2\left(\frac{\mu}{\lambda+\mu}\right)^2\sum_{k=a}^{b} q_2^{2k-2} + \frac{1}{\lambda}(\lambda+\mu-\mu q_2)\right\} - \left\{2\left(\frac{\mu}{\lambda+\mu}\right)^2\sum_{k=1}^{b} q_1^{2k-2} + \frac{1}{\lambda}(\lambda+\mu-\mu\, q_1)\right\}\right]E_1 \qquad (24)$$

where

$$A = \frac{c\mu q_1^2 - (\lambda + c\mu)q_1 + \lambda}{c\mu q_2^2 - (\lambda + c\mu)q_2 + \lambda}$$

and q_i $(i = 1,2)$ are the positive real roots of the equation.

$$c\mu\, z^{2b+2} - (\lambda + c\,\mu)z^2 + \lambda = 0 \qquad (25)$$

After finding the positive real roots q_1 and q_2 of the equation (25), the steady-state probabilities are computed from the expressions (23) and (24) for varying μ when λ is fixed. The numerical figures are tabulated in table : 6 and the related curves are presented in chart : 6.

Chart : 6 clearly shows that the steady-state probabilities for a general bulk service Erlangian queue with multi-server is less than that with single server when $n = 1,2,3,4$. This implies that the Erlangian queue with multi-server is more powerful than that with single server.

TABLE 1
STEADY STATE PROBABILITIES – P_N

System	ρ\n	0	1	2	3
M/M/1	0.1	0.9	0.09	0.009	0.0009
	0.2	0.8	0.16	0.032	0.0064
	0.25	0.75	0.19	0.047	0.0117
	0.4	0.6	0.24	0.096	0.0384
	0.5	0.5	0.25	0.125	0.0625
	0.6	0.4	0.24	0.144	0.0864
	0.7	0.3	0.21	0.147	0.1029
$E_2/E_2/1$	0.1	0.93	0.168	0.0027	0.000035
	0.2	0.73	0.2508	0.0142	0.000659
	0.25	0.69	0.2885	0.0243	0.001708
	0.4	0.55	0.3653	0.0709	0.012078
	0.5	0.46	0.3868	0.1121	0.029239
	0.6	0.37	0.3821	0.1542	0.057171
	0.7	0.28	0.3466	0.1858	0.092931
$HE_n/M/1$	0.1	0.9	0.0802	0.0159	0.0031
	0.2	0.8	0.1211	0.0478	0.0188
	0.25	0.75	0.1268	0.0625	0.0308
	0.4	0.6	0.0886	0.069	0.0537
	0.5	0.5	0.0353	0.0328	0.0305
	0.6	0.4	0.0129	0.0127	0.0124
	0.7	0.3	0.0071	0.0071	0.007

TABLE 2
NUMBER OF CUSTOMERS IN THE SYSTEM

ρ	M/M/1	$E_2/E_2/_1$	$HE_n/M/1$
0.1	0.111	0.1736	0.1247
0.2	0.25	0.2831	0.3304
0.25	0.333	0.3427	0.4927
0.4	0.667	0.5532	1.8051
0.5	1.000	0.7416	7.0922
0.6	1.500	0.0101	27.7778
0.7	2.333	1.4425	68.6275

TABLE 3
STEADY STATE PROBABILITIES – P_N FOR M/M/2

\	**0**	**1**	**2**	**3**
0.1	0.8182	0.1636	0.0164	0.0016
0.2	0.6667	0.2667	0.0533	0.0107
0.25	0.6000	0.3000	0.075	0.0187
0.4	0.4286	0.3429	0.1372	0.0549
0.5	0.3333	0.3333	0.1667	0.0833
0.6	0.2500	0.3000	1800	0.1080
0.7	0.1765	0.24271	0.173	0.1211

TABLE 4
NUMBER OF CUSTOMERS IN THE QUEUE

	M/M/1	**M/M/2**
0.1	0.0111	0.0020
0.2	0.0500	0.0167
0.25	0.0833	0.0333
0.4	0.2667	0.1524
0.5	0.5000	0.3333
0.6	0.9000	0.6750
0.7	1.6333	1.3453

TABLE 5
STEADY STATE PROBABILITIES P_N, (A = 2, B = 3)

n	**$M/M^{1,b}/2$**	**$M/M^{a,b}/1$**	**$M/M^{b}/1$**	**$M/M^{1,b}/1$**	**M/M/1**
0	0.075460	0.16116	0.21913	0.65740	0.83330
1	0.015194	0.05521	0.29421	0.22523	0.13890
2	0.003059	0.01892	0.31993	0.07716	0.02320
3	0.000616	0.00648	0.10959	0.02644	0.00390
4	0.000124	0.00222	0.03755	0.00906	0.00064
5	2.49E-05	0.00076	0.01286	0.00310	0.00011

TABLE 6
STEADY STATE PROBABILITIES – $P_{C,N}$, $-E_2/E_2^{A,B}/C$

c	\	**1**	**2**	**3**	**4**
1	0.06	0.3734	0.1809	0.0877	0.042500
	0.08	0.3474	0.1391	0.0557	0.022300
	0.10	0.2963	0.1015	0.0348	0.011900
	0.25	0.0689	0.0115	0.0019	0.000300
	0.40	0.0248	0.0028	0.0003	0.000340
	0.50	0.0150	0.0014	0.0001	0.000011
2	0.06	0.1658	0.04970	0.014900	0.00450000
	0.08	0.1242	0.02990	0.007200	0.00170000
	0.10	0.0871	0.01750	0.003500	0.00071000
	0.25	0.0193	0.00180	0.000160	0.00001500
	0.40	0.0078	0.00046	0.000027	0.00000160
	0.50	0.0051	0.00024	0.000012	0.00000055

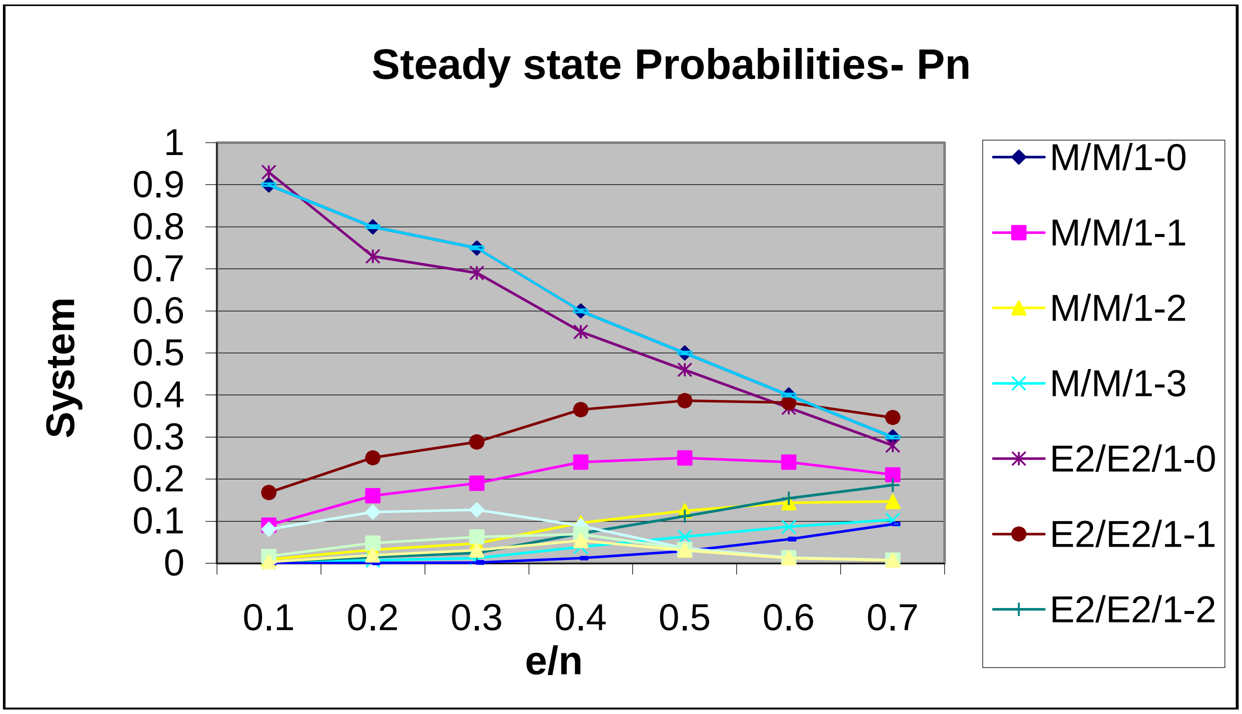

Fig.1 Chart – 1: Steady state probabilities - P_n

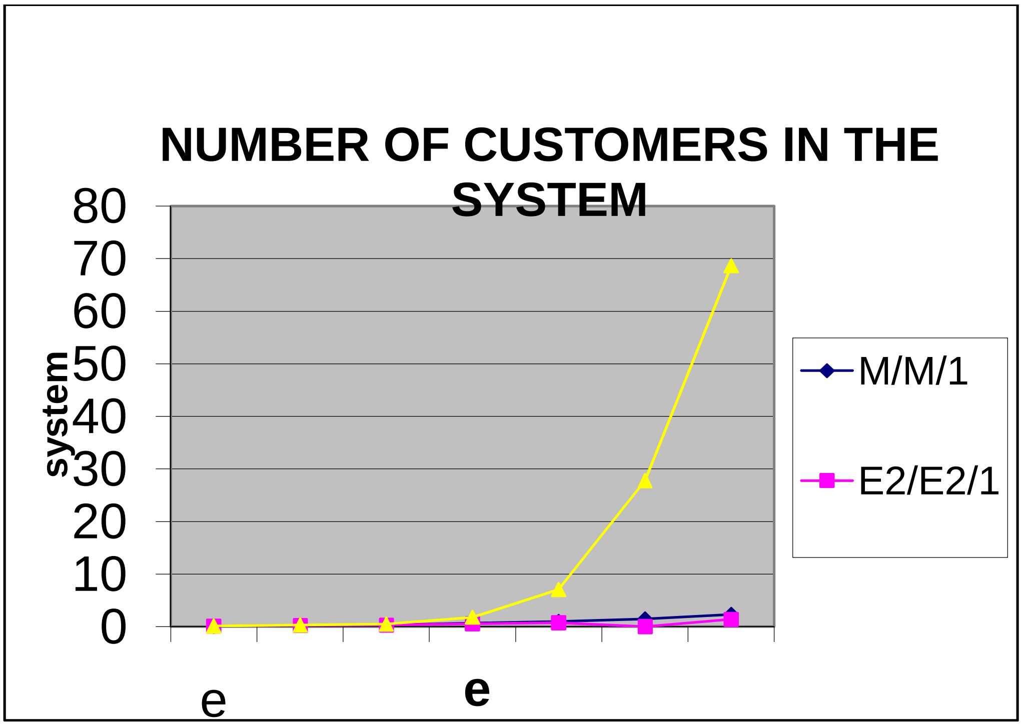

Fig.2 Chart – 2: Number of Customers in the System

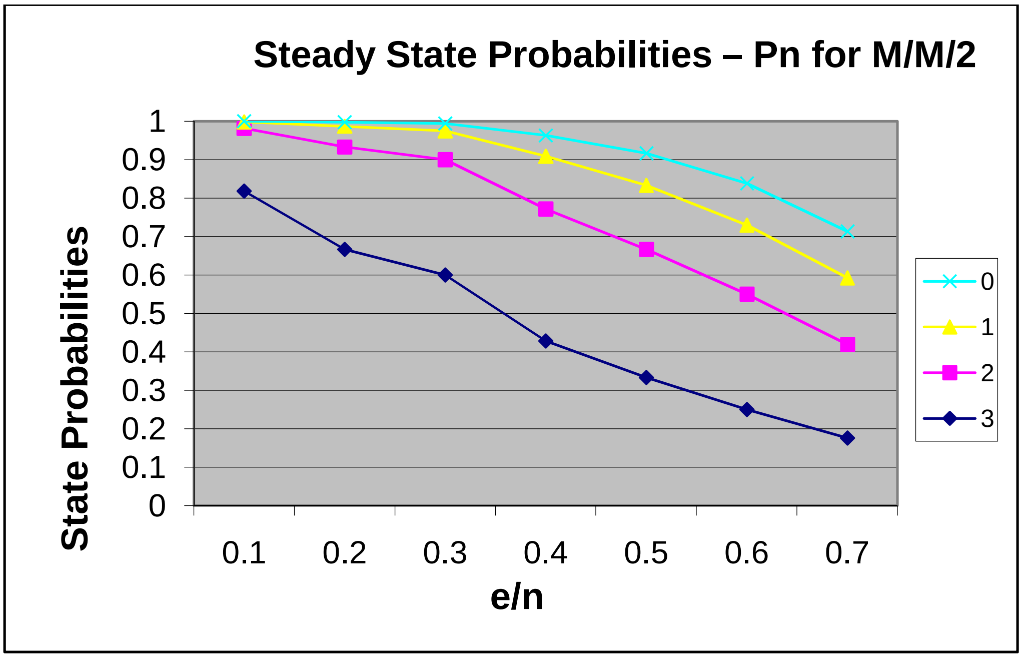

Fig.3 Chart – 3: Steady State Probabilities – Pn for M/M/2

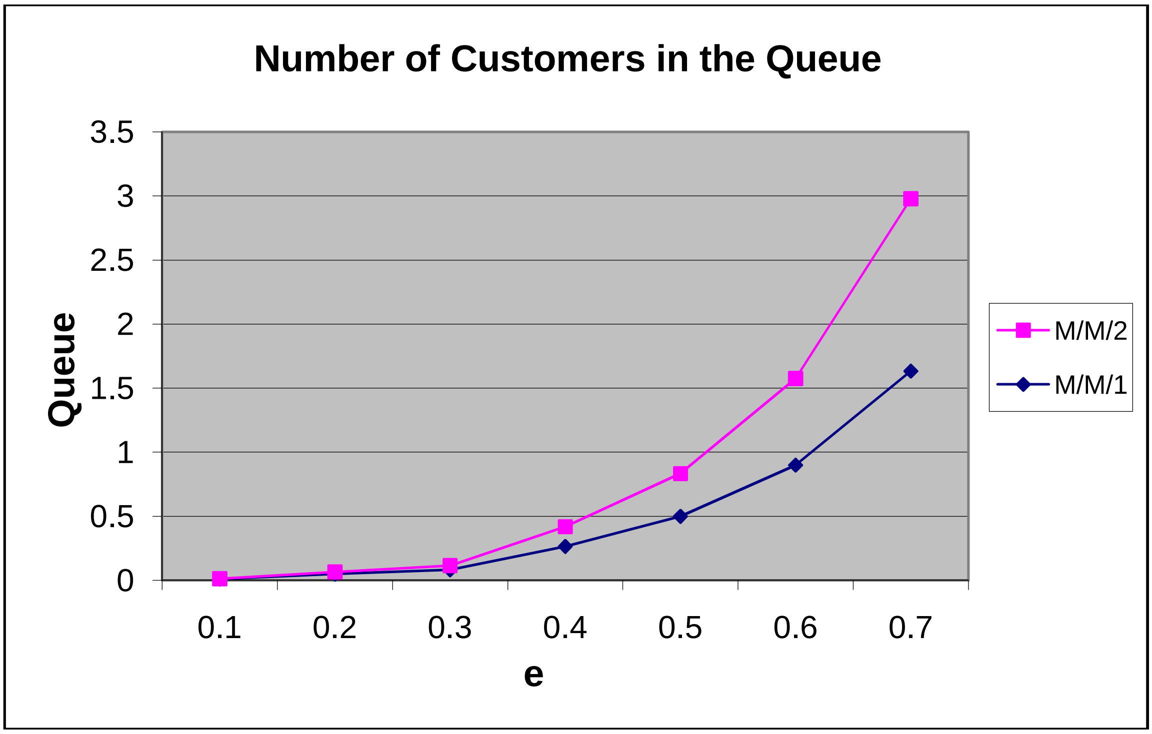

Fig. 4 Chart – 4: Number of Customers in the Queue

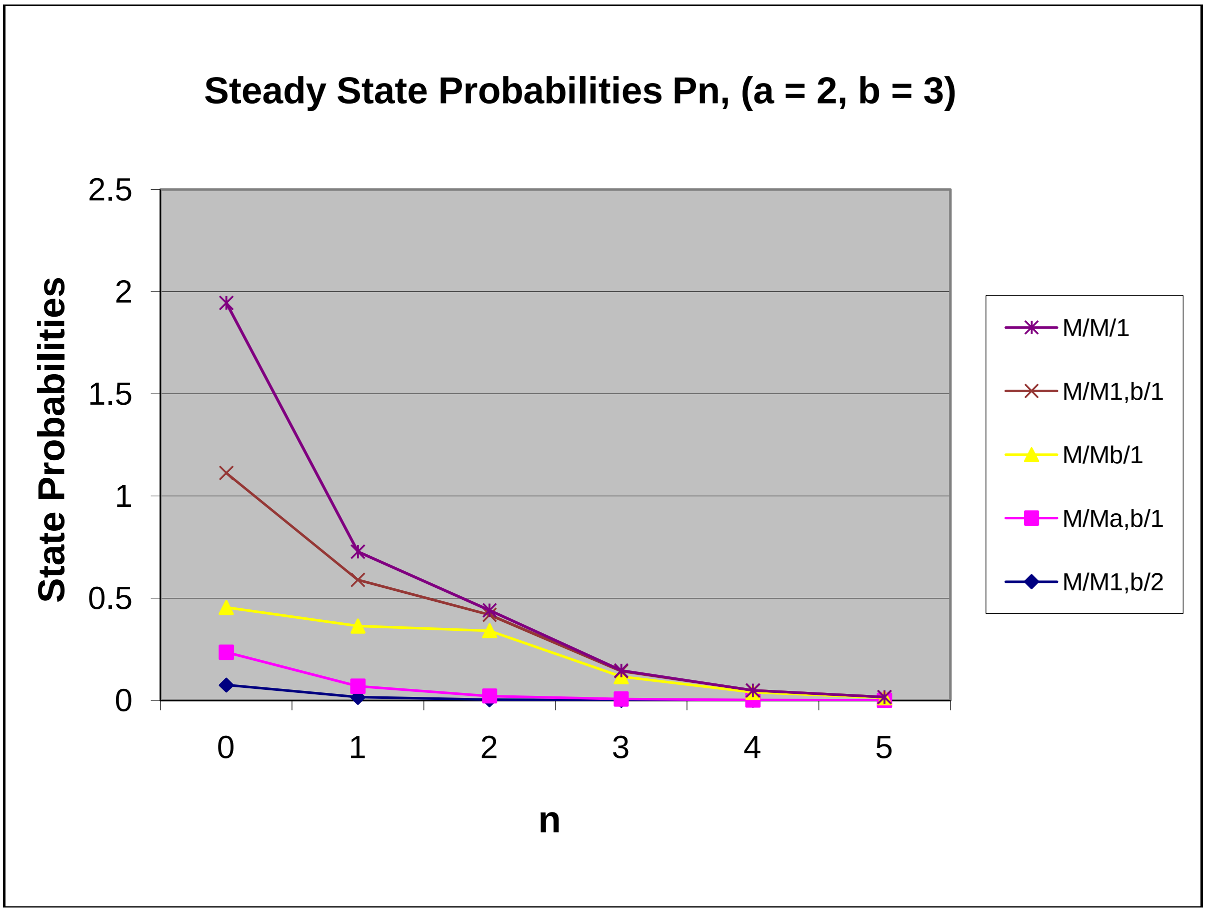

Fig.5 Chart – 5: Steady State Probabilities P_n, (a = 2, b = 3)

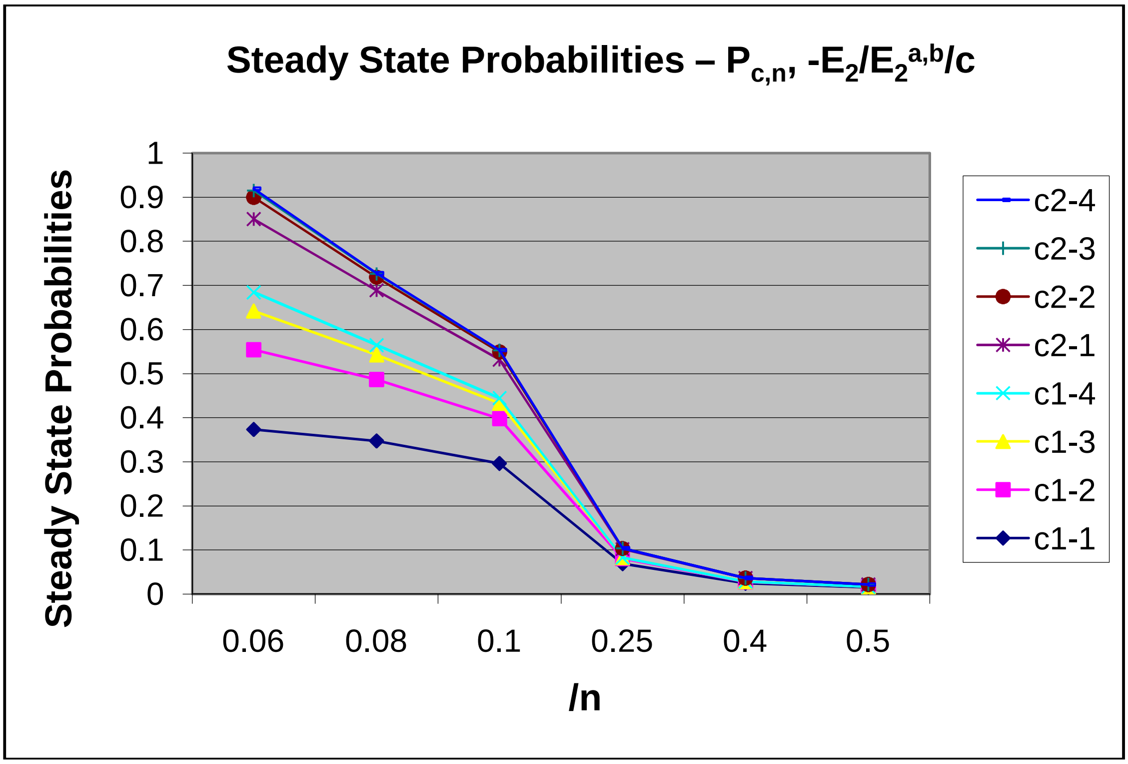

Fig.6 Chart – 6: Steady State Probabilities – $P_{c,n}$, $-E_2/E_2^{a,b}/c$

IV. CONCLUSION

When comparing M/M/1, and $HE_n/M/1$ models, second and third models are effective for lesser and higher values of ρ respectively.

Expected number of customers in $HE_n/M/1$ are rapidly increasing as compared with that of M/M/1 and $E_2/E_2/1$ for higher ρ. Single server is unable to control the queue $HE_n/M/1$ for higher ρ.

Single server Markovian queue is effective than multi server Markovian queue with respect to steady-state probabilities.

The expected number of customers waiting in a Markovian queue decrease as the number of servers increase.

The single server Markovian queue with general bulk service is effective than that with fixed and usual bulk services. On the other hand, the multi-server Markovian queue with usual bulk service is also effective than single server Markovian queues with single as well as bulk services.

The general bulk service Erlangian queue with multi server is more powerful than that with single server.

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