A Study on Triangular Graceful Graphs

S.P.RESHMA[1]

[1]Assistant Professor, Department of Mathematics, Emerald Heights College for women, Ooty, Tamil Nadu, India

spreshma30@gmail.com

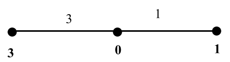
***Abstract:** A graph G with p vertices and q edges is said to be triangular graceful if there is an injective function φ : $V(G) \rightarrow X = \{0,1,2,\ldots T_q\}$, where T_q is the q^{th} triangular number. Define the function φ^* : $E(G) \rightarrow \{1,2,\ldots T_q\}$ such that $\varphi^*(u,v) = |\varphi(u) - \varphi(v)|$ for all edges (u,v). I $\varphi^*(E(G))$ is a sequence of distinct consecutive triangular numbers say $\{T_1,T_2,..,T_q\}$ then the function φ is said to be triangular. In this paper we prove the following graphs $S^+(n,m)$, Generalized Butane graph, n – Centipede union P_n, Fork graph are triangular graceful graphs.*

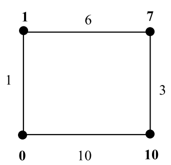
Keywords: **Star graph, Generalized Butane graph, Y-tree, n-centipede union P_n graph.**

Introduction:

In 1967 Rosa [4] introduced the β - valuation of a graph G. Golomb [3] subsequently called such labeling graceful. In 2001 Mr . Suresh Sing and Mr . Devaraj [1] call a graph G with p vertices and q edges **triangular graceful** if there is an injective function φ : V(G) → $X = \{0,1,2,\ldots T_q\}$, where T_q is the q^{th} triangular number. That is, $T_1 = 1, T_2 = 3$, $T_3 = 6, \ldots, T_n = \frac{n(n+1)}{2}$. Define the function φ^* : E(G) → $\{1,2,\ldots T_q\}$ such that $\varphi^*(u,v) = |\varphi(u) - \varphi(v)|$ for all edges (u,v). If $\varphi^*(E(G))$ is a sequence of distinct consecutive triangular numbers say $\{T_1,T_2,..,T_q\}$ then the function φ is said to be triangular graceful and the graph which admit such labeling is called a triangular graceful graph. In this paper we can see some classes of triangular graceful graphs.

Illustration : 1





P_3 is triangular graceful

C_4 is triangular graceful

SOME KNOWN RESULTS:

Mr . Suresh Sing and Mr . Devaraj proved the following results:

1. The path P_m is triangular graceful for all $m \geq 2$
2. The snark $K_{1,n}$ is triangular graceful for all $n \geq 1$
3. Olive trees are triangular graceful
4. Complete binary trees are triangular graceful
5. The star $S_{k,m}$ is triangular graceful
6. The double star S(m,n) , m ≥ 1, n ≥ 1 is triangular graceful
7. Caterpillars are triangular graceful
8. Cycles C_n are triangular graceful for $n \equiv 0$ (mod4)
9. Wheels W_n are not triangular graceful
10. The complete bipartite graph $K_{m,n}$ is not triangular graceful , for all m, n ≥ 2

Theorem:1.1

Let S_n be a star with n +1 vertices . Let G be the disjoint union of m copies of S_n. Then G is triangular graceful.

Proof

Let $\{a_0,a_1,a_2, \ldots,a_n\}$ be the vertices of the star S_n. Consider m isomorphic copies of S_n. Let G is the disjoint union of m copies of S_n.

Let $V(G) = \{a_{ij} / 1 \leq i \leq n + 1, 1 \leq j \leq m\}$.

Note that G has mn edges and m(n + 1) vertices.

Define f : V(G) → $\{0,1,2,..,T_{mn}\}$ as follows .

Now label the vertex $f(a_{11})$ as 0 and $f(a_{ij})$, j = 2,3,…,n+1as $T_{mn}, T_{m(n-1)}, \ldots, T_{mn-(n-1)}$ respectively so as the edges $f(a_{11}\ a_{1j})$, j = 2, 3,.., n + 1 must obtain the value as $T_{mn}, T_{mn-1}, T_{mn-2}, \ldots, T_{mn-(n-1)}$.

International Journal of Advanced and Innovative Research (2278-844)/

Volume 7 Issue 9

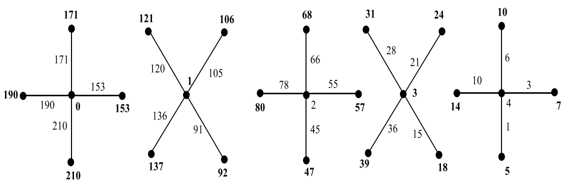
In the second copy, let a_{22}, .., $a_{2(n+1)}$ be the vertices adjacent to a_{21}. Label these vertices as $f(a_{21}) = 1$ and others as $T_{(mn-(n+1)-i)} + 1$, $1 \le i \le n$. So as the edges $f(a_{21}a_{2j})$, $2 \le j \le n + 1$ obtain the values by $f(a_{2j}) - f(a_{2i}) = T_{(mn-(n+1)-i)}$,$1 \le i \le n$.

Next let a_{32}, a_{33},…$a_{3(n+1)}$ be the vertices adjacent to a_{31} in the third copy. Label these vertices as $T_{mn-(2n+1-i)} + 2$, $1 \le i \le n$ and $f(a_{31}) = 2$ from this we obtain the edge labels as $f(a_{3j}) - f(a_{31}) = T_{mn-(2n+1-i)}$,$1 \le i \le n$.

Proceeding like this we get in the m^{th} copy of the graph G has the vertex set $a_{m1}, a_{m2}, \ldots, a_{m(n+1)}$. Labels the vertices as $f(a_{m1}) = m - 1$ and corresponding other vertices as $T_i + m - 1$, $i = 1,2,\ldots,n$.

Clearly all the vertex labelings are distinct and edge values are in the form $\{T_1, T_2, \ldots, T_{mn}\}$. This completes the proof. Hence G is triangular graceful.

Illustration : 2



5 copies of S_4 is triangular graceful.

Definition:1.1

Let S_n be a star with (n + 1) vertices. Consider m copies of S_n. Identify any one vertex of the i^{th} copy other than the central vertex with any one vertex other than the centre of $(i + 1)^{th}$ copy , the graph so obtained is denoted as $S^+(n,m)$.

Theorem:1.2

$S^+(n,m)$ is triangular graceful for all $n \ge 3$ and m.

Proof

Let $\{a_{ij} / 1 \le i \le n + 1, 1 \le j \le m\}$ be the vertex set of m copies of S_n. Then one vertex of the i^{th} copy other than the central vertex with any one vertex other than the centre of $(i + 1)^{th}$ copy.

Here we join the a_{in}^{th} vertex m copies to $a_{(i+1)n}^{th}$ vertices. The graph has (mn + 1) vertices and mn edges.

Define $f : V(G) \to \{0,1,2,..,T_{mn}\}$ as follows

$f(a_{11}) = 0$

$f(a_{1j}) = T_{mn-(n-(j-1))}$, $2 \le j \le n+1$

$f(a_{21}) = T_{mn-(n-1)} - T_{mn-n}$

$f(a_{2j}) = f(a_{21}) + T_{m+2n-(j+2)}$, $2 \le j \le n$

$f(a_{31}) = f(a_{2n}) - T_{(m-2)n}$

$f(a_{3j}) = f(a_{31}) - T_{(m-2)+n-j}$,$2 \le j \le n$

: : :

$f(a_{m1}) = f(a_{(m-1)n}) - T_n$

$f(a_{mj}) = f(a_{m1}) + T_{n-(j-1)}$, $2 \le j \le n$

Clearly the vertex labels are distinct.

Now from the definition, the edge values are

$| f(a_{1j}) - f(a_{11}) | = T_{mn-i}$, $0 \le i \le n - 1$, $j = i + 2$

$| f(a_{21}) - f(a_{1n}) | = T_{mn-n}$

$| f(a_{2j}) - f(a_{21}) | = T_{m+2n-(j+2)}$, $2 \le j \le n$

$| f(a_{31}) - f(a_{2n}) | = T_{(m-2)n}$

$| f(a_{3j}) - f(a_{31}) | = T_{(m-2)+n-j}$, $2 \le j \le n$

: : :

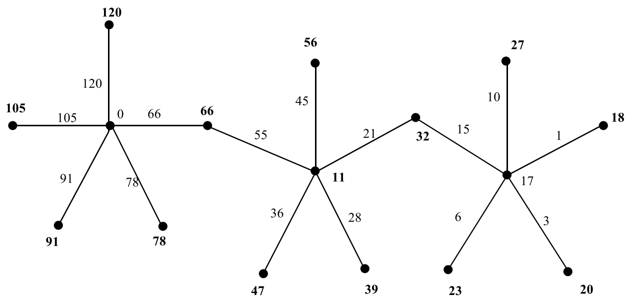
$| f(a_{m1}) - f(a_{(m-1)n}) | = T_n$

$| f(a_{m1}) - f(a_{mj}) | = T_{n-(j-1)}$, $2 \le j \le n$

Also, $| f(a_{m1}) - f(a_{mn}) | = T_1$

Hence the edge values are in the form $\{ T_1 , T_2, \ldots, T_{mn}\}$. Thus $S^+(n,m)$ is a triangular graceful graph .

Illustration : 3



S^+ (5,3) is triangular graceful .

Definition: 1.2 [Bull graph]

The bull graph is a planar undirected graph with 5 vertices and 5 edges in the form of a triangle with two disjoint pendant edges .

Theorem:1.3

Bull graph with one vertex attached with the root vertex is triangular graceful .

Proof

Let G be a bull graph with one vertex attached with the root vertex .
Let $V(G) = \{v_i / 1 \le i \le 6\}$ be the vertex set .
Then G has 6 vertices and 6 edges
Define $f : V(G) \to \{ 0,1,2,.. T_5\}$ as follows

$f(v_0) = 0$ $\quad f(v_1) = 15$
$f(v_2) = 21$ $\quad f(v_3) = 12$
$f(v_4) = 11$ $\quad f(v_5) = 1$

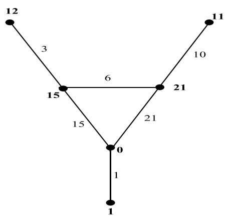
Clearly the vertex labels are distinct . Hence f is injective. It remains to show that the edge values are of the form $\{T_1, T_2, \dots,T_n\}$ Define the induced edge function $f^* :E(G) \to \{ 1,2,\dots, T_n\}$ by

$f^*(e_i) = | f(u_i) - f(v_i) |$ as

$f^*(e_1) = | f(v_1) - f(v_0) | = 15$
$f^*(e_2) = | f(v_2) - f(v_0) | = 21$
$f^*(e_3) = | f(v_1) - f(v_3) | = 3$
$f^*(e_4) = | f(v_4) - f(v_2) | = 10$
$f^*(e_5) = | f(v_1) - f(v_2) | = 6$
$f^*(e_6) = | f(v_5) - f(v_0) | = 1$

Clearly f^* is 1-1 and all the edges are of the form $\{ T_2,T_3 ,\dots,T_6\}$. Hence bull graph is triangular graceful.

Illustration: 4



Bull graph is triangular graceful .

Definition: 1.3 [Fork graph]

The fork graph sometimes also called the chair graph is the 5 vertices tree and it has 4 edges .

Theorem: 1.4

Fork graph is triangular graceful graph .

Proof

Let G be a fork graph with 5 vertices and 4 edges. Let the veretex set be $V(G) =\{ u_i /1 \le i \le 5\}$. Let the edge set be $E(G) = \{ u_i u_{i+1} / 1 \le i \le 2\} \cup \{u_1u_4\}\cup\{u_4u_5\}$. Define $f: V(G) \to\{ 0,1,2,..,T_4\}$ such that

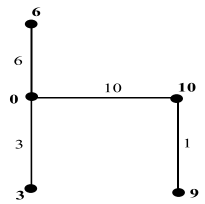
$f(u_1) = 0$; $f(u_2) = 3$; $f(u_3) = 6$
$f(u_4) = 10$; $f(u_5) = 9$

Clearly the vertex labels are distinct . Hence f is injective and the edge labels are of the form $\{T_1 , T_2,\dots,T_4\}$ is given by

$f^*(e_1) = 3 = T_2$ $\quad f^*(e_2) = 6 = T_3$
$f^*(e_3) = 10= T_4$ $\quad f^*(e_4) = 1 = T_1$

Clearly all the vertex labels are distinct and the edge labels are of the form $\{T_1,T_2,T_3, T_4\}$. Hence fork graph is triangular graceful graph.

Illustration: 5



Fork graph is triangular graceful graph.

Definition:1.4[Ladder rung graph]

Ladder rung graph is the graph union of n copies of the path graph P_2. It has 2n vertices and n edges.

Theorem: 1.5

Ladder rung graph is triangular graceful .

Proof

Let G be the ladder rung graph of 2n vertices and n edges .

Let $v_{i1},v_{i2},\dots v_{in}$, i = 1, 2 be the vertex set and $E(G) = \{v_{1j}v_{2j} / 1 \le j \le n\}$be the edge set of the graph. Define $f: V(G) \to \{0,1,2,..,T_n\}$ as follows

$f(v_{11}) = 0$ $\quad f(v_{i1}) = \sum_{j=1}^{i-1}(n-j)$, $2 \le i \le n$
$f(v_{12}) = T_n$ $\quad f(v_{i2}) = T_n - (i \square 1)$, $2 \le i \le n$.

Clearly all the vertex labels are distinct. Hence f is injective . It remains to show that the edge values

International Journal of Advanced and Innovative Research (2278-844)/

Volume 7 Issue 9

are of the form $\{T_1, T_2, \ldots, T_n\}$. Define the induced edge function

$f^*: E(G) \rightarrow \{1,2,..T_n\}$ as

$f^*(e_i) = | f(v_{i2}) - f(v_{i1})| = T_n - (i \,□\, 1) - \sum_{j=1}^{i-1}(n-j)$

$= T_n - (i\,□1) - [(n□1) + n\text{-}2 + \ldots + n□(i\,□\,1)]$

$= T_n - (i□1) - [(i\,□□1)n\,□\,\frac{(i-1)i}{2}]$

$= T_n - (i□1) - n(i□1) + \frac{i(i-1)}{2}$

$= \frac{n(n+1)}{2}\,□\,(i□1) - n(i□1) + \frac{i(i-1)}{2}$

$= [\, n(n+1)□2(i\,□1)□2n\,(i□1)\ i\,(i\,□\,1)\,] / 2$

$= [n^2 + n - (2n+2)(i\,□□1) + (i\,□□1)] / 2$

$= [n^2 + n\ □□(\,2ni\ □□2n + 2i\ □□2) + i2\ □□i\,]/2$

$= \frac{n^2 + 3n - 2ni - 3i + i^2 - 2}{2}$

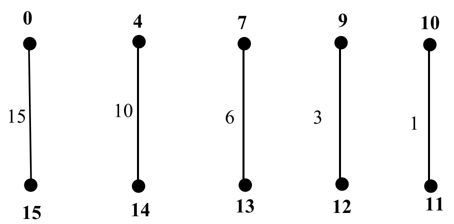
$= \frac{(n-i+1)(n-i+2)}{2} = \frac{[n-(i-1)][(n-i)+1]}{2}$

$f^*(e_i) = T_n - (i\,□\,1),\ i = 2,3,\ldots n$

$f^*(e_1) = | f(v_{11}) - f(v_{12}) | = T_n$

Clearly f^* is 1-1 and all the edges are of the form $\{T_1, T_2, \ldots, T_n\}$.Hence ladder rung graph is triangular graceful .

Illustration: 6



Ladder rung graph of $5P_2$ is triangular graceful.

Definition: 1.5 [Generalized Butane graph]

Generalized Butane graph is defined as follows. Let G be a graph with $V(G) = \{u_i / 1 \le i \le n\} \cup \{v_i / 1 \le i \le n\} \cup \{w_i / 0 \le i \le n+1\}$ and

$E(G) = \{u_i w_i / 1 \le i \le n\} \cup \{w_i v_i / 1 \le i \le n\} \cup \{w_i w_{i+1} / 0 \le i \le n\}$. Then the graph G has 3n + 2 vertices and 3n + 1 edges.

Theorem: 1.6

Generalized Butane graph is triangular graceful.

Proof

Let G be the graph with $V(G) = \{u_i / 1 \le i \le n\} \cup \{v_i / 1 \le i \le n\} \cup \{w_i / 0 \le i \le n + 1\}$ and $E(G) = \{u_i w_j / 1 \le i \le n\} \cup \{w_i v_i / 1 \le i \le n\} \cup \{w_i w_{i+1} / 0 \le i \le n\}$. Then the graph G has 3n + 2 vertices and 3n + 1 edges.

Define $f : V(G) \rightarrow \{0, 1, 2, 3, \ldots, T_{3n+1}\}$ as follows.

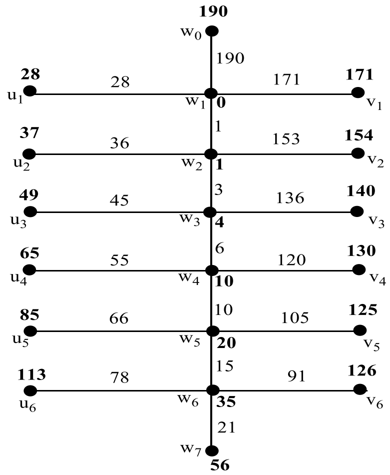
Now label the vertex $f(w_1)$ as 0 and $f(w_2) = T_1$; $f(w_3) = T_1 + T_2$, and $f(w_i) = T_{i-2} + T_{i-1}$, $4 \le i \le n + 1$. So as the edges $w_1w_2, w_2w_3, \ldots, w_nw_{n+1}$, must obtain the values as $T_1, T_2, \ldots, T_n$.

Next let $u_1, u_2, u_3, \ldots, u_n$ be the vertices adjacent to $w_1, w_2, w_3, \ldots, w_n$ in left. Label the vertices $f(u_i)$ as $T_{n+i} + f(w_i)$, $1 \le i \le n$ and so as the edges $u_1w_1, u_2w_2, \ldots, u_nw_n$ must obtain the values as T_{n+i}, $1 \le i \le n$.

Also let $v_1, v_2, v_3, \ldots, v_n$ be the vertices adjacent to $w_1, w_2, w_3, \ldots, w_n$ in right . Label the vertices $f(v_1)$ as T_{3n}, and $v_2, v_3, .., v_n$ as $T_{3n\,□□i} + f(w_{i+1})$, $1 \le i \le n\,□1$. And the corresponding edges $v_1w_1, v_2w_2, v_3w_3, \ldots, v_nw_n$ must obtain the values as $T_{3n\,□□i}$ for $0 \le i \le n\ □\ 1$.

Also the vertex $f(w_0)$ has 3n + 1 and the corresponding edge $f(w_0w_1) = 3n + 1$, Clearly all the vertex labelings are district and edge values are in the form $\{T_1, T_2, \ldots, T_{3n+1}\}$. This completes the proof. Hence Generalised Butane graph is triangular graceful.

Illustration: 7



Generalized Butane graph of n = 6 is triangular graceful.

Theorem: 1.7

n-centipede union P_n is triangular graceful.

Proof

The n-centipede is the tree on 2n nodes obtained by joining the bottoms of n copies of the path graph P_2 laid in a row with edges. It has 2n vertices and 2n – 1 edges. The path graph P_n is of n vertices and n – 1 edges.

Let G_1 be the n-centipede u_i and v_i, $1 \le i \le n$ and G_2 be the path P_n of $w_1, w_2, \ldots, w_n$.

Then $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. Now the graph G has 3n vertices and 3n – 2 edges.

Define $f : V(G) \to \{0, 1, 2, 3, \ldots, T_{3n-2}\}$ as follows.Now label the vertex as follows

Case (i) Suppose n is odd

$f(u_1) = 0 \quad f(u_2) = T_{3n-3}$

$f(u_{2i+1}) = f(u_{2i}) - T_{3n-2(i+1)}$ for $1 \le I \le \lceil \frac{n}{2} \rceil$

$f(u_{2i}) = f(u_{2i-1}) + T_{3n-(2i+1)}$ for $2 \le i \le \lfloor \frac{n}{2} \rfloor \square 1$

From this we get the edges $u_1u_2, u_2u_3, u_3u_4, \ldots, u_{n-1}u_n$ must obtain the values $T_{3n-3}, T_{3n-4}, \ldots, T_{2n-1}$. Also the vertex label of v_i are

$f(v_1) = T_{3n-2}$

$f(v_{2i+1}) = f(u_{2i+1}) + T_{2n-(2i+1)}$ for $1 \le i \le \lfloor \frac{n}{2} \rfloor$

$f(v_{2i}) = f(u_{2i}) \square T_{2n-2i}$ for $1 \le i \le \lceil \frac{n}{2} \rceil$

Case (ii) suppose n is even

$f(u_1) = 0 \quad f(u_2) = T_{3n-3}$

$f(u_{2i+1}) = f(u_{2i}) - T_{3n-2(i+1)}$ for $1 \le i \le \frac{n}{2}$

$f(u_{2i}) = f(u_{2i-1}) + T_{3n-(2i+1)}$ for $2 \le i \le \frac{n}{2} \square 1$

From this we get the edges $u_1u_2, u_2u_3, u_3u_4, \ldots, u_{n-1}u_n$ must obtain the values $T_{3n-3}, T_{3n-4}, \ldots, T_{2n-1}$. Also the vertex label of v_i are

$f(v_1) = T_{3n-2}$

$f(v_{2i+1}) = f(u_{2i+1}) + T_{2n-(2i+1)}$ for $1 \le i \le \frac{n}{2}$

$f(v_{2i}) = f(u_{2i}) \square T_{2n-2i}$ for $1 \le i \le \frac{n}{2}$

So as from above results the edge u_1v_1 must obtain the value T_{3n-2}, and the remaining edges $u_2v_2, u_3v_3, \ldots, u_nv_n$ must obtain the values $T_{2n-2}, T_{2n-3}, \ldots, T_n$.

Also the vertex label of w_i, $1 \le i \le n$ by

$f(w_n) = 1$

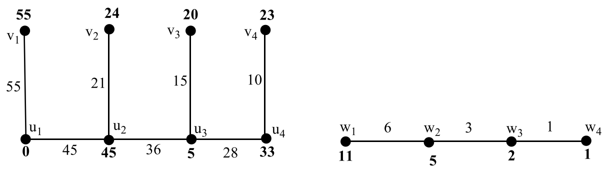
$f(w_{n \square 1}) = 2$

$f(w_{n \square i}) = f(w_{n \square (i-1)}) + T_i$, $2 \le i \le n \square 1$

and so the edges w_iw_{i+1}, $1 \le i \le n - 1$ must obtain the value $T_{n-1}, T_{n-2}, T_{n-3}, \ldots, T_1$.

Clearly all the vertex labels are distinct and the edge values are in the form $\{T_1, T_2, \ldots, T_{3n-2}\}$. This complete the proof. Hence G is triangular graceful.

Illustration: 8



4-centipede union P_4 is triangular graceful.

Theorem: 1.8

$K_{1,n} \cup K_2$ is triangular graceful.

Proof

International Journal of Advanced and Innovative Research (2278-844)/

Volume 7 Issue 9

Let $K_{1,n}$ of u_0 , u_1, . . . , u_n vertices and v_1, v_2 be the vertices of K_2.

Note that the graph $K_{1,n} \cup K_2$ has n + 3 vertices and n + 1 edges.

Define f : V(G) →{0, 1, 2, . . . ,T_{n+1}} as follows

$f(u_0) = 0 \quad f(u_i) = T_i$, $2 \le i \le n + 1$

$f(v_1) = 4 \quad f(v_2) = 5$

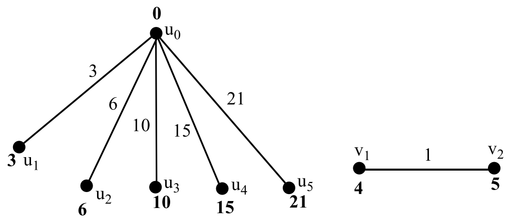
Clearly all the vertex labels are distinct. Hence f is injective the edge labels are of the form {T_1, T_2, T_3, . . . , T_{n+1}} by

$f^*(e_i) = | f(u_0) - f(u_i) | = T_i$, $2 \le i \le n + 1$

$f^*(e_1) = | f(v_1) - f(v_2) | = T_1$

Thus all the edge labels are of the form {T_1, T_2, T_3, . . . , T_{n+1}}. This completes the proof. Hence $K_{1,n} \cup K_2$ is triangular graceful.

Illustration:9



$K_{1,5} \cup K_2$ is triangular graceful.

Definition:1.6

Y- tree is the tree obtained by taking three paths of same length and identifying one point of each path .

Theorem: 1.9

Y-tree is triangular graceful for all n.

Proof

Let V(G) = {u_1, u_2, . . . , u_n} ∪ {v_1, v_2, . . . , v_n}∪{w_1, w_2, . . . , w_n}where $u_1 = v_1 = w_1$ and E(G) = {u_iu_{i+1} /$1 \le i \le n - 1$} ∪{v_iv_{i+1} / $1 \le i \le n - 1$} ∪ {w_iw_{i+1} /$1 \le i \le n - 1$ }.

Note that the graph Y- tree has 3n – 2 vertices and 3(n – 1) edges.Define f : V(G) → {0, 1, 2, . . . , $T_{3(n-1)}$} as follows

Case (a) Suppose n is odd

$f(u_1 = v_1 = w_1) = 0$

$f(u_2) = T_{n-1}$

$f(u_{2i+1}) = f(u_{2i})$ □□ $T_{n □ 2i}$, for $1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$

$f(u_{2i}) = f(u_{2i-1}) + T_{n-(2i-1)}$, for $2 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$

$f(u_n) = f(u_{n-1}) + T_1$; $f(v_2) = T_{2n-2}$

$f(v_{2i}) = f(v_{2i-1}) + T_{2n-2i}$, for $2 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$

$f(v_{2i+1}) = f(v_{2i}) - T_{2n-(2i+1)}$, for $1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$

$f(w_2) = T_{3(n-1)}$

$f(w_{2i+1}) = f(w_{2i}) - T_{3n-(2i+2)}$, $1 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$

$f(w_{2i}) = f(w_{2i-1}) + T_{3n-(2i+1)}$, $2 \le i \le \left\lfloor \frac{n}{2} \right\rfloor$

Case (b) Suppose n is even

$f(u_1 = v_1 = w_1) = 0$

$f(u_2) = T_{n-1}$

$f(u_{2i+1}) = f(u_{2i})$ □ T_{n-2i}, for $1 \le i \le \frac{n}{2}$ □ 1

$f(u_{2i}) = f(u_{2i-1}) + T_{n-(2i-1)}$, for $2 \le i \le \frac{n}{2}$

$f(v_2) = T_{2n-2}$

$f(v_{2i}) = f(v_{2i-1}) + T_{2n-2i}$, for $2 \le i \le \frac{n}{2}$

$f(v_{2i+1}) = f(v_{2i}) - T_{2n-(2i+1)}$,for $1 \le i \le \frac{n}{2}$ □ 1

$f(w_2) = T_{3(n-1)}$

$f(w_{2i+1}) = f(w_{2i}) - T_{3n-(2i+2)}$, $1 \le i \le \frac{n}{2}$ □ 1

$f(w_{2i}) = f(w_{2i-1}) + T_{3n-(2i+1)}$, $2 \le i \le \frac{n}{2}$

Clearly all the vertex labels are district. Hence f is injective. It remains to show that the edge values are of the form {T_1, T_2, T_3, . . . , $T_{3(n-1)}$}. Define the induced edge function $f^□$: E(G) → {1,2,..,$T_{3(n-1)}$} by

$f^□$ (u(e_i)) = f(u_iu_{i+1}) = T_{n-i} , $1 \le i \le n$ □1

$f^□$ (v(e_i)) = f(v_iv_{i+1}) = $T_{2n-(i+1)}$, $1 \le i \le n$ □ 1

$f^□$ (w(e_i)) = f(w_iw_{i+1}) =$T_{3n-(i+2)}$,$1 \le i \le n$□□ 1

Clearly f^* is 1 -1 and f^*(E(G)) ={T_1,T_2,….,$T_{3(n1)}$}.This completes the proof. Hence Y-tree is triangular graceful.

Illustration: 10

Let us verify the algorithm for n = 8. Then G is as follows

$f(u_1 = v_1 = w_1) = 0$

$f(u_2) = T_{n-1} = T_7 = 28$

$f(u_{2i+1}) = f(u_{2i})$ □ T_{n-2i}, for $1 \le i \le \frac{n}{2}$ □ 1

$f(u_{2i}) = f(u_{2i-1}) + T_{n-(2i-1)}$, for $2 \le i \le \frac{n}{2}$

$f(u_3) = f(u_2)$ □ $T_{8-2} = f(u_2)$ □ $T_6 = 28$ □ $21 = 7$

$f(u_4) = f(u_3)$ □ $T_5 = 7 + 15 = 22$

$f(u_5)$ □ $f(u_4)$ □ $T_4 = 22 - 10 = 12$

$f(u_6) = f(u_5)$ □ $T_3 = 12 + 6 = 18$

$f(u_7) = f(u_6)$ □ $T_2 = 18$ □ $3 = 15$

$f(u_8) = f(u_7)$ □ $T_1 = 15 + 1 = 16$

$f(v_2) = T_{2n-2} = T_{16-2} = T_{14} = 105$

$f(v_{2i}) = f(v_{2i-1}) + T_{2n-2i}$, for $2 \le i \le \frac{n}{2}$

$f(v_{2i+1}) = f(v_{2i}) - T_{2n-(2i+1)}$, for $1 \le i \le \frac{n}{2}$ □

1

$f(v_3) = f(v_2) - T_{16-(2+1)} = f(v_2) - T_{13} = 105 - 91 = 14$

$f(v_4) = f(v_3) + T_{16-4} = 14 + T_{12} = 14 + 78 = 92$

$f(v_5) = f(v_4)$ □ $T_{11} = 92$ □ $66 = 26$

$f(v_6) = f(v_5)$ □ $T_{10} = 26 + 55 = 81$

$f(v_7) = f(v_6)$ □ $T_9 = 81$ □ $45 = 36$

$f(v_8) = f(v_7)$ □ $T_8 = 36 + 36 = 72$

$f(w_2) = T_{3(n-1)} = T_{3(8-1)}$

$f(w_2) = T_{21} = 231$

$f(w_{2i+1}) = f(w_{2i}) - T_{3n-(2i+2)}$, $1 \le i \le \frac{n}{2}$ □ 1

$f(w_{2i}) = f(w_{2i-1}) + T_{3n-(2i+1)}$, $2 \le i \le \frac{n}{2}$

$f(w_3) = f(w_2) - T_{24-4} = 231 - T_{20} = 231 - 210 = 21$

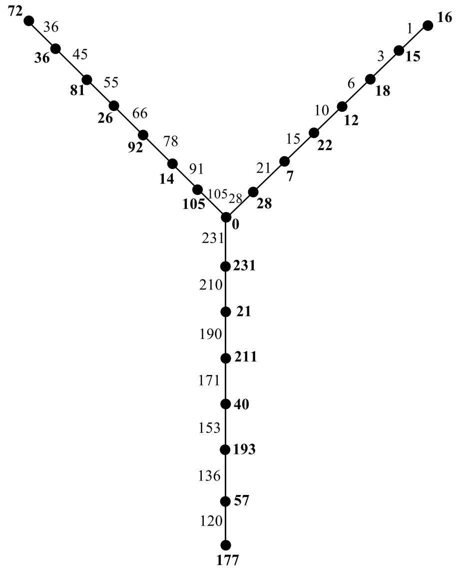
$f(w_4) = f(w_3) + T_{24-5} = 21 + T_{19} = 21 + 190 = 211$

$f(w_5) = f(w_4)$ □ $T_{18} = 211$ □ $171 = 40$

$f(w_6) = f(w_5) + T_{17} = 40 + 153 = 193$

$f(w_7) = f(w_6) - T_{16} = 193 - 136 = 57$

$f(w_8) = f(w_7) + T_{15} = 57 + 120 = 177$



Y-tree is triangular graceful when n = 8.

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