On Total Domination Polynomial in Graphs

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***Abstract*: In this paper we find out the total domination polynomial of Herschel graph, Bull graph , Wagners graph and for some connected graphs such as H(n) , KS(n), $C_4\hat{o}K_{1,n}$, $G''(n), CK(n), G_*((n), J(n), C(n)$.**

***Keywords*: Total domination polynomial, Herschel graph, Bull graph, Wagners graph, connected graphs.**

I. Introduction

Unless mentioned or otherwise, a graph in this paper shall mean a simple finite graph and without isolated vertices. We denote the vertex set and edge set of the graph G by V(G) and E(G) respectively . For standard terminology and notations we follow [3]. A set S of vertices in a graph G is said to be a dominating set if every vertex u ϵ V is either an element of S or is adjacent to an element of S . A set of vertices in a graph G is said to be a total dominating set if every vertex u ϵ V is adjacent to an element of S . In([4], [5]) , we shall see the total domination polynomial of paths and cycles**.** Now ,we shall find out the total domination polynomial of some graphs.

Definition 1:

Let G be a simple graph with no isolated vertices. Let $\mathfrak{D}_t(G,i)$ be the family of total dominating sets of G with cardinality i and let $d_t(G,i) = |\mathfrak{D}_t(G,i)|$. Then,the total domination polynomial $D_t(G,x)$ of G is defined as $D_t(G,x) = \sum_{i=\gamma_t(G)}^{n} d_t(G,i)x^i$,where $\gamma_t(G)$ is the total domination number of G.

II. Herschel graph [2]

Definition 2:

The Herschel graph is a bipartite graph with 11 vertices and 18 edges , the smallest non –Hamiltonian polyhedral graph.

Theorem 1:

The total domination polynomial of the Herschel graph G is $D_t(G,x) = 23x^4 + 133x^5 + 279x^6 + 281x^7 + 160x^8 + 54x^9 + 11x^{10} + x^{11}$

Proof :

Let G be the Herschel graph with 11 vertices and 18 edges . Let the vertex set of G be V(G)={1,2,3,4,5,6,7,8,9,10,11}.The graph G is shown in the figure 1.

For G,the total domination number is $\gamma_t(G) = 4$.

The number of total dominating sets of G with cardinality 4,5,6,7,8,9,10,11.

$$d_t(G,4) = |\mathfrak{D}_t(G,4)| = 23$$
$$d_t(G,5) = |\mathfrak{D}_t(G,5)| = 133$$
$$d_t(G,6) = |\mathfrak{D}_t(G,6)| = 279$$
$$d_t(G,7) = |\mathfrak{D}_t(G,7)| = 281$$
$$d_t(G,8) = |\mathfrak{D}_t(G,8)| = 160$$
$$d_t(G,9) = |\mathfrak{D}_t(G,9)| = 54$$
$$d_t(G,10) = |\mathfrak{D}_t(G,10)| = 11$$
$$d_t(G,11) = |\mathfrak{D}_t(G,11)| = 1$$

Therefore $D_t(G,x) = \sum_{i=\gamma_t(G)}^{n} d_t(G,i)x^i$

$= d_t(G,4)x^4 + d_t(G,5)x^5 + d_t(G,6)x^6 + d_t(G,7)x^7 + d_t(G,8)x^8 + d_t(G,9)x^9 + d_t(G,10)x^{10} + d_t(G,11)x^{11}$

$= 23x^4 + 133x^5 + 279x^6 + 281x^7 + 160x^8 + 54x^9 + 11x^{10} + x^{11}$

Hence, the total domination polynomial of the Herschel graph G is given as

$D_t(G,x) = 23x^4 + 133x^5 + 279x^6 + 281x^7 + 160x^8 + 54x^9 + 11x^{10} + x^{11}$.

Illustration 1:

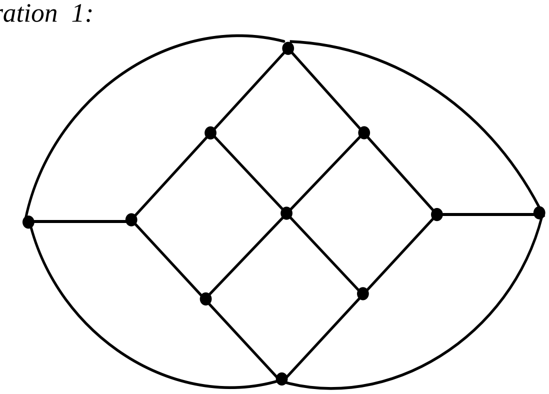

Fig 1 Herschel graph

III. Bull graph

Definition 3:

The Bull graph is a planar undirected graph with 5 vertices and 5 edges in the form of a triangle with two disjoint pendant edges.

Theorem 2:
The total domination polynomial of the bull graph B is given by $D_t(B,x) = x^2(x+1)^3$.
Proof:
Let B be the bull graph with 5 vertices and 5 edges.
Let the vertex set of B be {1,2,3,4,5} as shown in the figure 2 .
For B,the total domination number $\gamma_t(B) = 2$.
The number of total dominating sets of B with cardinality 2,3,4,5 is given below:

$$d_t(G,2) = |\mathfrak{D}_t(G,2)| = 1$$
$$d_t(G,3) = |\mathfrak{D}_t(G,3)| = 3$$
$$d_t(G,4) = |\mathfrak{D}_t(G,4)| = 3$$
$$d_t(G,5) = |\mathfrak{D}_t(G,5)| = 1$$

The total domination polynomial of the Bull graph B is

$$D_t(B,x) = x^2 + 3x^3 + 3x^4 + x^5$$
$$= x^2(1 + 3x + 3x^2 + x^3)$$
$$= x^2(x+1)^3$$

Therefore $D_t(B,x) = x^2(x+1)^3$
This is true for all n ≥1.

Illustration 2:

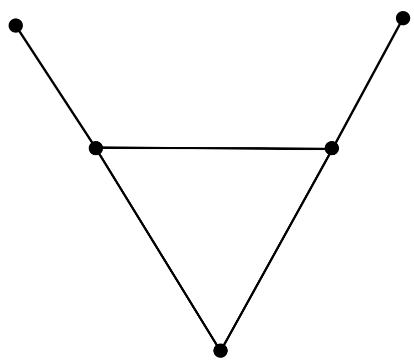

Fig 2 Bull Graph

IV. Wagner's graph

Definition 4:
The Wagner graph is a 3 – regular graph with 8 vertices and 12 edges . It is the 8 – vertex Mobius ladder graph .
Theorem 3:
The total domination polynomial of the Wagners graph W is given by

$$D_t(W,x) = x^8 + 8x^7 + 28x^6 + 48x^5 + 38x^4 + 8x^3$$

Proof :
Let W be the Wagners graph with 8 vertices and 12 edges.
Let the vertex set of W be {1,2,3,4,5,6,7,8} as shown in the figure 3.
For W,the total domination number is $\gamma_t(W) = 3$.
The number of total dominating sets of W with cardinality 3,4,5,6,7,8,9,10,11.

$$d_t(W,3) = |\mathfrak{D}_t(W,3)| = 8$$
$$d_t(W,4) = |\mathfrak{D}_t(W,4)| = 38$$
$$d_t(W,5) = |\mathfrak{D}_t(W,5)| = 48$$
$$d_t(W,6) = |\mathfrak{D}_t(W,6)| = 28$$
$$d_t(W,7) = |\mathfrak{D}_t(W,7)| = 8$$
$$d_t(W,8) = |\mathfrak{D}_t(W,8)| = 1$$

Therefore $D_t(W,x) = \sum_{i=\gamma_t(G)}^{n} d_t(W,i)x^i$

$$= d_t(W,3)x^3 + d_t(W,4)x^4 + d_t(W,5) x^5 + d_t(W,6)x^6 + d_t(W,7)x^7 d_t(W,8)x^8$$
$$= 8x^3 + 38x^4 + 48x^5 + 28x^6 + 8x^7 + x^8$$

Hence ,the total domination polynomial of the Wagners graph W is given as

$$D_t(W,x) = 8x^3 + 38x^4 + 48x^5 + 28x^6 + 8x^7 + x^8.$$

Illustration 3:

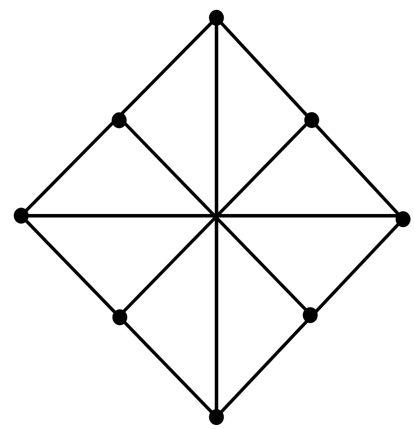

Fig 3 Wagners graph

V. Connected graphs

Theorem 4 :
The total domination polynomial of the connected graph $G_*(n) = P_n + \overline{K_2}$ is $D_t(G_*(n), x) = x^2(x+1)^{2n}$ for $n \geq 2$.
Proof:
Let $G = G_*((n)$.
Let $\{u_i / 1 \leq i \leq n\}$ be the vertices of G.
Let us add $\overline{K_2}$ to P_n .
The resultant graph obtained is $G^*(n)$.
It has $3n$ vertices and $3n - 1$ edges .
Hence

$$D_t(G,x) = x^2\left(\binom{2n}{0} + \binom{2n}{1}x + \binom{2n}{2}x^2 + \cdots + \binom{2n}{2n}x^{2n}\right)$$
$$= x^2(x+1)^{2n}$$

This is true for all for $n \geq 2$.

Note 1:
The total domination polynomial of the connected graph $G = G_*(n)$ is given below :

1. $D_t(G_*(2),x) = x^6 + 4x^5 + 6x^4 + 4x^3 + x^2$
2. $D_t(G_*(3),x) = x^8 + 6x^7 + 15x^6 + 20x^5 + 15x^4 + 6x^3 + x^2$
3. $D_t(G_*(4),x) = x^{10} + 8x^9 + 28x^8 + 56x^7 + 70x^6 + 56x^5 + 28x^4 + 8x^3 + x^2$

Ilustration 4 :

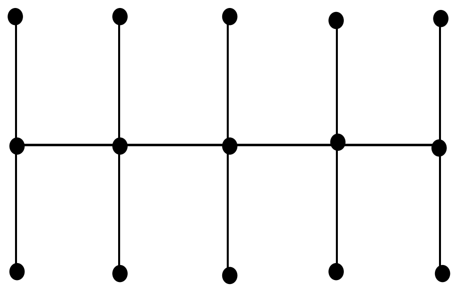

Fig 4 $G_*(5)$.

Theorem 5 :

The total domination polynomial of the connected graph J(n) is $D_t(J(n), x) = x^2(x+1)^{n+2}$ for $n \geq 1$.

Proof:

Let J(n) be the connected graph with $n+4$ vertices and $2n+3$ vertices .

Let $V(J(n)) = \{u_i/1 \leq i \leq 4\} \cup \{a_i/1 \leq i \leq n\}$ and $E(J(n)) = \{u_i u_{i+1}/1 \leq i \leq 3\} \cup \{u_2 a_i/1 \leq i \leq n\}\} \cup \{u_3 a_i/1 \leq i \leq n\}$.

The resultant graph so obtained is J(n).

$$D_t(G,x) = x^2\left(\binom{n+2}{0} + \binom{n+2}{1}x + \binom{n+2}{2}x^2 + \cdots + \binom{n+2}{n+2}x^{n+2}\right)$$
$$= x^2(x+1)^{n+2}$$

This is true for all $n \geq 1$.

Illustration 5:

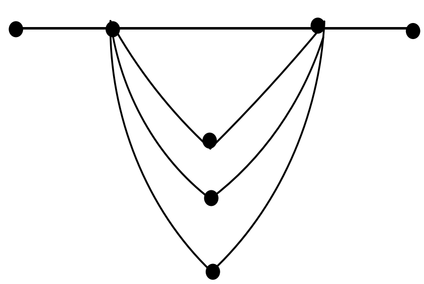

Fig 5 J(3)

Note 2:

The total domination polynomial of the connected graph $J(n)$ is given below :

1. $D_t(J(1), x) = x^5 + 3x^4 + 3x^3 + x^2$
2. $D_t(J(2), x) = x^6 + 4x^5 + 6x^4 + 4x^3 + x^2$
3. $D_t(J(3), x) = x^7 + 5x^6 + 10x^5 + 10x^4 + 5x^3 + x^2$
4. $D_t(J(4), x) = x^8 + 6x^7 + 15x^6 + 20x^5 + 15x^4 + 6x^3 + x^2$
5. $D_t(J(5), x) = x^9 + 7x^8 + 21x^7 + 35x^6 + 35x^5 + 21x^4 + 7x^3 + x^2$

Theorem 6:

The total domination polynomial of the connected graph C(n) is $D_t(C(n), x) = x^2(x+1)^{2n+1}$ for $n \geq 1$.

Proof :

Let $G = C(n)$ be the connected graph with $2n+3$ vertices and edges .

Let (u, v, x) be the vertices of C_3.

Let $V(G) = V(C_3) \cup \{a_i, b_i/1 \leq i \leq n\}$ and $E(G) = E(C_3) \cup \{ua_i/1 \leq i \leq n\} \cup \{vb_i/1 \leq i \leq n\}$.

Hence

$$D_t(G,x) = x^2\left(\binom{2n+1}{0} + \binom{2n+1}{1}x + \binom{2n+1}{2}x^2 + \cdots + \binom{2n+1}{2n+1}x^{2n+1}\right)$$
$$= x^2(x+1)^{2n+1}$$

This is true for all $n \geq 1$.

Illustration 6:

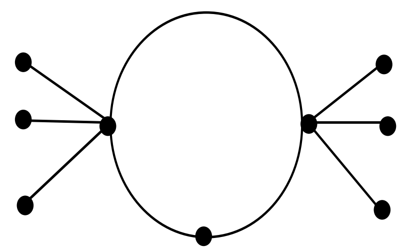

Fig 6 C(3)

Note 3:

The total domination polynomial of the connected graph $G = C(n)$ is given below :

1. $D_t(C(1), x) = x^5 + 3x^4 + 3x^3 + x^2$
2. $D_t(C(2), x) = x^7 + 5x^6 + 10x^5 + 10x^4 + 5x^3 + x^2$
3. $D_t(C(3), x) = x^9 + 7x^8 + 21x^7 + 35x^6 + 35x^5 + 21x^4 + 7x^3 + x^2$
4. $D_t(C(4), x) = x^{11} + 9x^9 + 36x^8 + 84x^7 + 126x^6 + 84x^5 + 36x^4 + 9x^3 + x^2$

Theorem 7 :

The total domination polynomial of the connected graph $G = H(n)$ is $D_t(H(n), x) = x^2[(x+1)^{n+1}(x+3)]$ for $n \geq 1$.

Proof :

Let $G = H(n)$ be the connected graph with $n+4$ vertices and $2n+4$ vertices.

Let (u,v,x,y) be the vertices of C_4.

Let $V(G) = V(C_4) \cup \{a_i/1 \leq i \leq n\}$ and

$E(G) = E(C_4) \cup \{ua_i/1 \leq i \leq n\} \cup \{va_i/1 \leq i \leq n\}$.

$$D_t(G,x) = x^2\left[\left(\binom{n+1}{0} + \binom{n+1}{1}x + \binom{n+1}{2}x^2 + \cdots + \binom{n+1}{n+1}x^{n+1}\right)(x+3)\right]$$
$$= x^2[(x+1)^{n+1}(x+3)]$$

This is true for all $n \geq 1$.

Illustration 7:

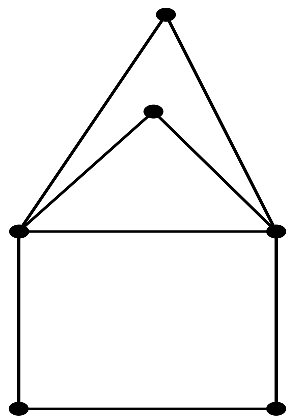

Fig 7 $H(2)$

Note 4:

The total domination polynomial of the connected graph $G = H(n)$ is given below :

1. $D_t(H(1),x) = x^5 + 5x^4 + 7x^3 + 3x^2$
2. $D_t(H(2),x) = x^6 + 6x^5 + 12x^4 + 10x^3 + 3x^2$
3. $D_t(H(3),x) = x^7 + 7x^6 + 18x^5 + 22x^4 + 13x^3 + 3x^2$
4. $D_t(H(4),x) = x^8 + 8x^7 + 25x^6 + 40x^5 + 35x^4 + 16x^3 + 3x^2$

Theorem 8:

The total domination polynomial of the connected graph $G = KS(n)$ is $D_t(G,x) = x^3(x+1)^{n+2}$ for $n \geq 1$.

Proof :

Let $G = KS(n)$ be the connected graph with n+5 vertices and n+8 vertices.

Let (u,v,x,y) be the vertices of K_4.

Let $V(G) = V(K_4) \cup \{a\} \cup \{b_i/1 \leq i \leq n\}$ and $E(G) = E(K_4) \cup \{ua, va\} \cup \{ab_i/1 \leq i \leq n\}$.

Then the total domination polynomial of G is

$$D_t(G,x) = x^3\left(\binom{n+2}{0} + \binom{n+2}{1}x + \binom{n+2}{2}x^2 + \cdots + \binom{n+2}{n+2}x^{n+2}\right) = x^3(x+1)^{n+2}$$

This is true for all $n \geq 1$.

Illustration 8:

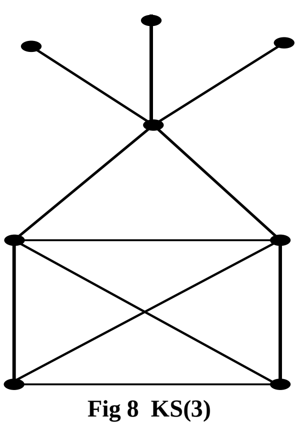

Fig 8 KS(3)

Note 5:

The total domination polynomial of the connected graph $G = KS(n)$ is given below:

1. $D_t(KS(1),x) = x^6 + 3x^5 + 3x^4 + x^3$
2. $D_t(KS(2),x) = x^7 + 4x^5 + 6x^6 + 4x^4 + x^3$
3. $D_t(KS(3),x) = x^8 + 5x^7 + 10x^6 + 10x^5$
4. $+5x^4 + x^3$
5. $D_t(KS(4),x) = x^9 + 6x^8 + 15x^7 + 20x^6 + 15x^5 + 6x^4 + x^3$
6. $D_t(KS(5),x) = x^{10} + 7x^9 + 21x^8 + 35x^7 + 35x^6 + 21x^5 + 7x^4 + x^3$

Theorem 9 :

The total domination polynomial of the connected graph $G = C_4\hat{o}K_{1,n}$ is given by

$$D(G,x) = x^2(x+1)^{n+1}(x+2) \text{ for } n \geq 1.$$

Proof :

Let u_1, u_2, u_3, u_4 be the vertices of C_4.

Let $V(G) = \{u_i/1 \leq i \leq 4\} \cup \{v_i/1 \leq i \leq n\}$ be the vertex set of G and the edge set of G is $E(G) = \{u_iu_{i+1}/1 \leq i \leq 4\} \cup \{u_4u_1\} \cup \{v_iu_2/1 \leq i \leq n\}$.

The resultant graph so obtained is $G = C_4\hat{o}K_{1,n}$.

G consists of n+4 vertices and edges.

Then the total domination polynomial of the connected graph $C_4\hat{o}K_{1,n}$ is given by $D(G,x) = x^2(x+1)^{n+1}(x+2)$

This is true for all $n \geq 1$.

Illustration 9:

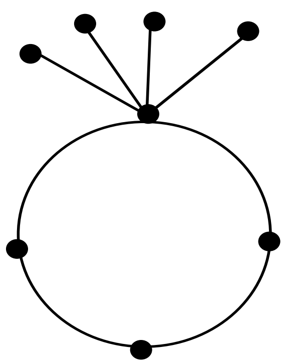

Fig 9 $C_4\hat{o}\, K_{1,4}$

Note 6:

The first few total domination polynomial of the connected graph $G = C_4\hat{o}K_{1,n}$ is given below :

1. $D(C_4\hat{o}K_{1,1}, x) = 2x^2 + 5x^3 + 4x^4 + x^5$
2. $D(C_4\hat{o}K_{1,2}, x) = 2x^2 + 7x^3 + 9x^4 + 5x^5 + x^6$
3. $D(C_4\hat{o}K_{1,3}, x) = 2x^2 + 9x^3 + 16x^4 + 14x^5 + 6x^6 + x^7$
4. $D(C_4\hat{o}K_{1,5}, x) = 2x^2 + 11x^3 + 25x^4 + 30x^5 + 20x^6 + 7x^7 + x^8$

Theorem 10:
The total domination polynomial of the connected graph which obtained from path graph P_3 attached with n copies of K_1 at one end is $D_t(G''(n),x) = x^2(x+1)^{n+1}$ for $n \geq 1$.
Proof:
Let $G''(n)$ be the connected graph with $n+3$ vertices and $n+2$ vertices .
Let $V(G''(n)) = \{u_i/1 \leq i \leq 3\} \cup \{a_i/1 \leq i \leq n\}$ and
$E(G''(n)) = \{u_i u_{i+1}/1 \leq i \leq 2\} \cup \{u_3 a_i/1 \leq i \leq n\}$.
The resultant graph so obtained is $G''(n)$.

$$D_t(G''(n),x) = x^2\left(\binom{n+1}{0} + \binom{n+1}{1}x + \binom{n+1}{2}x^2 + \cdots + \binom{n+1}{n+1}x^{n+2}\right)$$
$$= x^2(x+1)^{n+1}$$

This is true for all $n \geq 1$.

Illustration 10 :

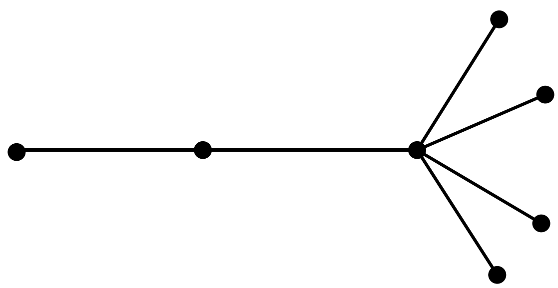

Fig 10 $G''(4)$

Note 7:
The total domination polynomial of the connected graph $G''(n)$ is given below :

1. $D_t(G''(1),x) = x^2 + 2x^3 + x^4$
2. $D_t(G''(2),x) = x^2 + 3x^3 + 3x^4 + x^5$
3. $D_t(G''(3),x) = x^2 + 4x^3 + 6x^4 + 4x^5 + x^6$
4. $D_t(G''(4),x) = x^2 + 5x^3 + 10x^4 + 10x^5 + 5x^6 + x^7$

Theorem 11 :
The total domination polnomial of the connected graph $CK(n)$ is given by
$D_t(CK(n),x) = x^2(x+1)^{n+2} + x^3(x+1)^n - x^3$ for $n \geq 1$.
Proof :
Let C_3 be the cycle graph with vertices u_1, u_2, u_3.
Let $K_{1,n}$ be the star with vertices $v_0, v_1, v_2, \ldots, v_n$.
Let us join C_n and $K_{1,n}$ by an edge $\{u_3 v_0\}$.
The resultant graph thus obtained is $CK(n)$ which shown in the figure.
It has $n+4$ vertices and $n+4$ edges.
The total domination polynomial of the connected graph $CK(n)$ is given by
$D_t(CK(n),x) = x^2(x+1)^{n+2} + x^3(x+1)^n - x^3$
This is true for all $n \geq 1$.

Illustration 11:

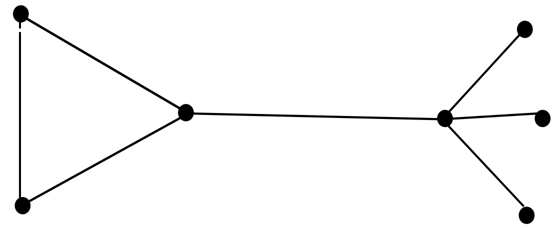

Fig 11 $CK(3)$

Note 6:
The total domination polynomial of the graph $CK(n)$ is given below :

1. $D_t(CK(1),x) = x^5 + 4x^4 + 3x^3 + x^2$
2. $D_t(CK(2),x) = x^6 + 5x^5 + 8x^4 + 4x^3 + x^2$
3. $D_t(CK(3),x) = x^7 + 6x^6 + 13x^5 + 13x^4 + 5x^3 + x^2$
4. $D_t(CK(4),x) = x^8 + 7x^7 + 19x^6 + 26x^5 + 19x^4 + 6x^3 + x^2$

VI . ACKNOWLEDGEMENT

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