Copairable Graphs

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***Abstract* - Let G(V,E) be a connected graph. For a vertex v, in V the set of all adjacent vertices of v is called an open neighbourhood of v and is denoted by N(v). The neighbourhood complement of v is denoted by $N(v)^c$, the set of all non-adjacent vertices of v along with v itself. The closed neighbourhood of v is defined by N[v] = N(v) $\cup$ {v}. Any two adjacent vertices u and v in G is said to be pairable, if N[u] = N[v]. A copairable graph is defined as a graph in which for any vertex u $\in$ V, there exists a vertex v in V, such that a vertex w in V is adjacent to u if and only if it is not adjacent to v. In this paper, we study about this new family of graphs.**

***Keywords:* Pairable vertices, pairable graphs, copairable vertices, copairable graphs.**

AMS Subject Classification Code (2010): 05C (Primary)

I. Introduction

Throughout this paper, we consider only finite, simple, undirected and connected graphs. For notations and terminology, we follow [4]. Let G(V,E) be a graph of order n. For any vertex v $\in$ V, the *open neighbourhood* of v is the set of all vertices adjacent to v and is denoted by N(v). The *neighbourhood complement* of v is denoted by $N(v)^c$, which is defined as $N(v)^c$ = V(G) - N(v). The *closed neighbourhood* of v is the set of all vertices adjacent to v along with itself and is denoted by N[v], that is N[v] = N(v) $\cup$ {v}. A vertex of degree n-1 is called a *full vertex*. A graph G is said to be *r - regular* if degree of every vertex in G is of degree r. *Path* on n vertices is denoted by P_n and *cycle* on n vertices is denoted by C_n .

The *distance* d(u,v) between any two vertices u and v is the length of a shortest path between them. The *eccentricity* [5] e(u) of a vertex u is the distance of a farthest vertex from u. The *radius* rad(G) of G is the minimum eccentricity and the *diameter* diam(G) of G is the maximum eccentricity of the graph G. A vertex u with e(u) = rad(G) is called a *central vertex*. A graph G for which rad(G) = diam(G) is called a *self- centered graph*.

A graph is said to be a *unicyclic graph* if it has exactly one cycle. A *Bistar* $B_{m,n}$ is obtained from $K_{1,m}$ and $K_{1,n}$ by joining the centres of them by means of an edge. Any graph which contains no C_3 as a subgraph is called a *triangle free graph*.

A *dominating set* [6] is a subset S of the vertex set V such that every vertex is either in S or adjacent to a vertex in S, that is, such that every vertex in V-S is adjacent to at least one vertex in S. A dominating set S is called a *minimum dominating* set if there is no dominating set S' in G such that $|S'| < |S|$. The *dominating number* γ(G) of G is the number of vertices in a minimum dominating set of G.

The graph S(G), obtained from G, by adding a new vertex w for every vertex v $\in$ V and joining w to all vertices of G adjacent to v, is called the *splitting graph* [7] of G. For example, a graph G and its splitting graph S(G) are shown in Figure 1.

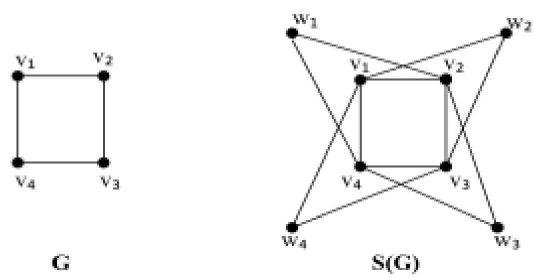


Figure 1

Let G be a graph with vertex set {v_1, v_2, …, v_n}. The *cosplitting graph* [2] CS(G) is the graph obtained from G, by adding a new vertex w_i for each vertex v_i and joining w_i to all vertices which are not adjacent to v_i in G. For example, a graph G and its cosplitting graph CS(G) are shown in Figure 2.

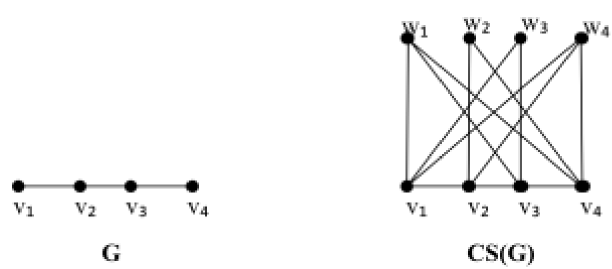


Figure 2

Let G be a graph with vertex set V(G) = {v_1, v_2,… v_n} the *2 - splitting graph* [1] S_2(G) of G is defined as the graph with vertex set V(S_2(G)) = {u_1, u_2,…u_n,w_1,w_2,…w_n} and edge set E(S_2(G)) = {u_iu_j, w_iw_j, u_iw_j / $v_iv_j \in$ E(G), 1 ≤ i, j ≤ n}. For example, a graph G and its 2 - splitting graph $S_2(G)$ are shown in Figure 3.

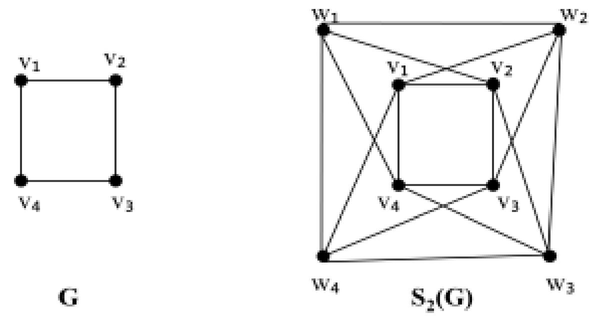


Figure 3

A connected graph G is said to be *neighbourhood highly irregular (NHI)* [8], if any two distinct vertices in the open neighbourhood of v, have distinct closed neighbourhood sets, that is, for any vertex v, if $u,w \in N(v)$, $(u \neq w)$, then $N[u] \neq N[w]$. For more results on NHI graphs, one can refer [3].

For any two distinct vertices u and v in G, u is said to be *pairable* [1] with v if N[u] = N[v] in G. A vertex in G is called a *pairable vertex* if it is pairable with a vertex in G. Clearly any two pairable vertices are adjacent and have the same degree. A connected graph G of order at least 2 is said to be a *pairable graph*, if every vertex of G is pairable. For example, K_n is a pairable graph of order n for any $n \geq 2$. It has been proved in [1], that a graph G is NHI if and only if it contains no pairable vertices. This forces that pairable graphs are not NHI.

A *1-factor* is a 1 – regular spanning subgraph of G. A 1 – factor F in a pairable graph is said to be a *pairing 1 –factor* if E(F) ={uv∈ E(G) / u is pairable with v in G}. For example, a pairable graph G, a pairing 1- factor (shown in bold lines) are given in Figure 4.

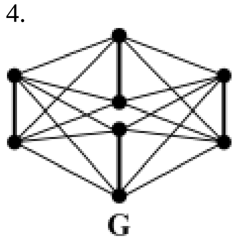


Figure 4

Result 1[1] Let G be a pairable graph of order 2n, $n \geq 1$. Then G – F is a 2 – splitting graph if and only if F is a pairing 1 – factor.

In this paper, we introduce a new concept of copairable vertices and copairable graphs. For any two adjacent vertices u and v in G, u is said to be copairable with v if $N(u) = N(v)^c$. A vertex in G is said to be *copairable vertex* if it is copairable with a vertex in G. The set of all copairable vertices of G is denoted by CP(G).

A connected graph G of order at least 2 is said to be a *copairable graph* if every vertex of G is copairable. For example, $K_{n,m}$ is a copairable graph of order m+n for any $n,m \geq 1$. This proves the existence of copairable graphs of given order $n \geq 2$.

In this paper, we obtain a few results on copairable graphs.

II. Main Results

Define a relation ρ on CP(G) by $u \rho v$ if and only if $N(u) = N(v)^c$. Then clearly ρ is symmetric on CP(G).

The following facts can be easily verified for copairable graphs:

Fact 1 For any graph G, CP(G) cannot be a singleton set.

Fact 2 Any full vertex in a graph G is copairable with a pendant vertex.

Fact 3 Let G be a graph for which $CP(G) \neq \phi$ and $\Delta(G) = n-1$, then G contains only one full vertex.

For, if G contains two full vertices u and v, then neither u nor v is a copairable vertex, since $\delta(G) \geq 2$. In addition if a non full vertex w is copairable with a vertex x, then $N(w) \cap N(x)$ contains both u and v, which is a contradiction. Therefore, G contains only one full vertex.

Fact 4 The central vertices of a bistar are copairable vertices.

Fact 5 For a path P_n, $CP(P_n) \neq \phi$ if and only if $2 \leq n \leq 4$.

Fact 6 A copairable graph has a full vertex if and only if $G \cong K_{1,n}$.

Fact 7 If $CP(G) \neq \phi$, for a graph G, then $\gamma(G) \leq 2$.

Fact 8 If copairable vertices exist in a graph G, then $diam(G) \leq 3$ and $rad(G) \leq 2$.

For, let u and v be two copairable vertices in a graph G. Then $N(u) \cup N(v) = V(G)$ and so the eccentricity of u and v is at most 2. But the eccentricity of any vertex other than u and v is at most 3. Then $diam(G) \leq 3$ and $rad(G) \leq 2$.

Fact 9 A copairable graph with $\Delta(G) \neq n-1$ is self-centered with radius 2.

For, let G be a copairable graph in which $\Delta(G) \neq n-1$. Then G contains no full vertex. Therefore $e(v) \geq 2$ for any vertex v in G. Also by Fact 8, if u is a copairable vertex, then $e(u) \leq 2$. But in G every vertex is copairable and hence $e(v) \leq 2$ for any vertex v in G. This forces that $e(v) = 2$ for any vertex v in G. Thus G is self centered with radius 2.

Fact 10 A copairable graph G is a tree if and only if G is a star.

For, let G be a copairable graph which is a tree. Then G contains at least two pendant vertices. By Fact 2, pendant vertex is copairable with a full vertex. But a tree contains a full vertex if and only if it is a star. Therefore G is a star. And the converse is obvious.

The following theorems establish some properties of copairable vertices in a graph G.

Theorem 1 Let $G \ncong K_{1,n}$ be any graph with copairable vertices u and v. If uv is a bridge, then u and v are the only central vertices of G.

Proof Let G be a graph with copairable vertices u and v. Assume that uv is a bridge. Since $G \ncong K_{1,n}$ by Fact 9, rad(G) = 2. If G contains an edge joining a neighbour of u and v, then uv lies on a cycle, which is a contradiction. Hence eccentricity of any vertex other than u and v is 3. Therefore every edge of G other than uv belongs to either $\langle N(u) \rangle$ and $\langle N(v) \rangle$. Since G contains no full vertex, neither u nor v is a pendant vertex. Every neighbour of u is at distance three to every neighbour of v and $e(u) = e(v) = 2$. Hence u and v are the only central vertices of G. ■

Theorem 2 Pairable graph G is copairable if and only if $G \cong K_2$.

Proof Let G be a pairable graph which is copairable. Suppose $G \ncong K_2$. Let u be a pairable vertex in G. Then there is a vertex v in G such that N[u] = N[v]. Then every neighbour of u is also a neighbour of v. Hence for any $w \in N(v)$, we have $v \in N(u) \cap N(w)$, that is, there does not exist a vertex $w \in N(v)$ such that $N(u) \cap N(w) = \phi$, since $v \in N(u) \cap N(w)$. Hence u is copairable with v alone. This is possible only when $G \cong K_2$. And the converse is obvious. ■

Let $C_r(v)$ denote the cycle $v_1v_2\ldots v_rv_1$ of order r with a fixed vertex $v=v_1$ and let $C_r(v)(m_1,m_2,\ldots m_r)$, where $m_i \geq 0$, be the graph obtained from $C_r(v)$ by identifying the central vertex of the star K_{1,m_i} with the vertex v_i of $C_r(v)$, for $1 \leq i \leq r$. Note that we take $K_{1,0}$ as K_1 vacuously. Let $C'_3(v)(m_1,m_2)$, $m_i \geq 0$, be the graph obtained from $C_3(v)$ by identifying the vertex of degree m_1+1 in B_{m_1,m_2} at v. For example, $C_3(v)(2,3,0)$ and $C'_3(v)(2,3)$ are shown in Figure 5.

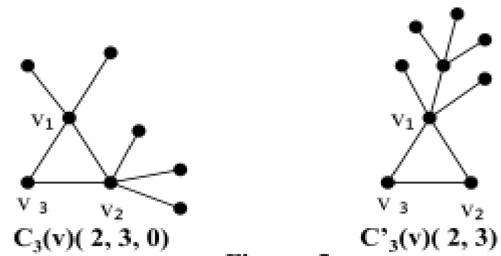


Figure 5

Theorem 3 Let G be any graph with $CP(G) \neq \phi$. Then G is unicyclic if and only if $G \cong C_4(v)(m_1,m_2,0,0)$ for $m_1,m_2 \geq 0$ or $C'_3(v)(m_1,m_2)$ for $m_2 \geq 0$.

Proof Let G be any unicyclic graph with copairable vertices u and v.

Case (i) Suppose the edge uv lies in the unique cycle. Since u and v have no common neighbour and every vertex of G is a neighbour of u or v, the unique cycle is of length 4. Since G is unicyclic there is no edge between any two neighbours of u and that of v. The resultant graph is isomorphic to $C_4(v)(m_1,m_2,0,0)$ for $m_1,m_2 \geq 0$.

Case (ii) Suppose the edge uv is not in the unique cycle. Then uv is a bridge. Now either N(u) contains every vertex of the unicycle or N(v) contains every vertex of the unicycle. Since every edge in N(u) or N(v) induces C_3 in G, we conclude that $\langle N(u) \rangle \cup \langle N(v) \rangle$ contains at most one edge. The resultant graph is isomorphic to $C'_3(v)(m_1,m_2)$ for $m_2 \geq 0$. The converse is obvious. ■

Corollary 4 C_4 is the unique unicyclic copairable graph.

Proof Let G be unicyclic copairable graph. Then by previous theorem, $CP(G) \cong C_4(v)(m_1,m_2,0,0)$ for $m_1,m_2 \geq 0$ or $C'_3(v)(m_1,m_2)$ for $m_2 \geq 0$. In the above graph, CP(G) = V(G) if and only if $G \cong C_4$. Hence C_4 is the only unique unicyclic copairable graph. ■

Theorem 5 If u is a copairable vertex in G, then u cannot be a copairable vertex of $\overline{G}$.

Proof Let u be a copairable vertex in G. Then there exists a vertex $v \in V(G)$ such that $N(u) = N(v)^c$ in G. Assume that $u \in CP(\overline{G})$. Clearly $uv \notin E(\overline{G})$. Then there exists a vertex w in $\overline{G}$ such that $N(u) = N(w)^c$ in $\overline{G}$. Hence $uw \in E(\overline{G})$ and $w \notin N(u)$ in G. Then $w \in N(v)$ which implies that $vw \notin E(\overline{G})$. Now $v \notin N$

$_{\overline{G}}(u) \cup N_{\overline{G}}(w)$ which is a contradiction. Therefore u cannot be a copairable vertex of $\overline{G}$. ■

Corollary 6 If G is a copairable graph, then $\overline{G}$ is not a copairable graph. ■

Let P(G) denote the set of paendant vertices of G.

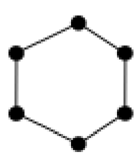
Lemma 7 Let G be a graph for which CP(G) $\neq \phi$, δ(G) = 1 and Δ(G) = n-1. Then G contains only one full vertex v and CP(G) = {v} $\cup$ P(G).

Proof Let G be a graph for which CP(G) $\neq \phi$, δ(G) = 1 and Δ(G) = n-1. Then by Fact 3, G contains only one full vertex v. If u is a pendant vertex in G, then N(u) $\cup$ N(v) = V(G) and N(u) $\cap$ N(v) = ϕ. Thus any pendant vertex is copairable with v. Conversely, let w be a vertex in G such that 1 < d(w) < n-1. If w is copairable with a vertex x in G, then 1< d(x) < n-1. But N(w) $\cap$ N(x) contains the vertex v, which is a contradiction. Therefore, w is not a copairable vertex. Hence CP(G) = {v} $\cup$ P(G). ■

Theorem 8 Let G be a graph with δ(G) >1. If CP(G) $\neq \phi$, then γ(G) =2.

Proof Let G be a graph with no pendant vertices. Let CP(G) ≠ ϕ. Let u and v be two copairable vertices of G. Then u and v have no common neighbour. By Fact 6, Δ(G) < n-1, then γ(G) ≠1. If u is copairable with v, then N(u) $\cup$ N(v) = V(G) and thus S = {u,v} is a minimal dominating set of G. Hence γ(G) = 2. ■

Note that the converse of the above theorem need not be true. For example, the graph G shown in Figure 6 has γ(G) = 2 but CP(G) = ϕ.



G
Figure 6

Theorem 9 Let G be any triangle free n – regular graph of order 2n. Then G is a copairable graph.

Proof Let G be any triangle free n – regular graph of order 2n. Let u and v be any two adjacent vertices in G. Since G is a triangle free graph, then N(u) $\cap$ N(v) = ϕ. Also $|N(u) \cup N(v)| = |N(u)| + |N(v)| - |N(u) \cap N(v)|$ = n + n - 0 = 2n = $|V(G)|$. Therefore N(u) $\cup$ N(v) =V(G). Hence u and v are copairable. ■

Note that the converse of the above theorem need not be true. For example, the complete bipartite graph $K_{m,n}$, m $\neq$ n is a copairable graph but it is not regular.

Theorem 10 Every graph G is an induced subgraph of a graph H in which copairable vertices are the central vertices.

Proof Let G be a graph. If G itself has copairable vertices as central vertices, then there is nothing to prove. Suppose not, construct a graph H with vertex set V(H) =V(G) $\cup$ {u,v,w} and the edge set E(H) =E(G) $\cup$ {uv,vw,uu$_i$ / u$_i \in$ V(G)}. By our construction, G is an induced subgraph of H. Clearly u and v are the only copairable vertices in H with eccentricity 2. All other vertices are of eccentricity 3 in H. Hence u and v are the only central vertices of H. For example, a graph G and a graph H in which G is an induced subgraph and copairable vertices are central vertices are shown in Figure 7. ■

G H
Figure 7

Theorem 11 Any copairable graph other than star has C_4 as an induced subgraph.

Proof Let G $\ncong K_{1,n}$ be a copairable graph. Let u be a copairable vertex which is copairable with v in G. If d(u) =1, then v is a full vertex. Therefore, by Fact 6, G $\cong K_{1,n}$ which is a contradiction. Since every vertex of G is copairable, none of them can be a pendant vertex and so δ(G) ≥2. Then there exist two vertices w ≠ v and x ≠ u in G such that w$\in$N(u), x$\in$N(v). Since u,v $\in$ CP(G), x$\notin$N(u) and w$\notin$N(v).

Case (i) w is copairable with u in G. But u is copairable with v and so, N(w) =N(v). This forces that x$\in$N(w). Now N(u) $\cap$ N(v) = ϕ. Therefore vw, ux $\notin$E(G). Hence the subgraph induced by the vertices u, w, x and v is C_4.

Case (ii) w is not copairable with u. Then assume that w is copairable with y $\neq$ u. Therefore N(w) = $N(y)^c$. Then y $\notin$ N(u), since if y $\in$ N(u), then N(y) $\cap$ N(w) = {u} $\neq \phi$. Hence uy $\notin$ E(G). Since u is copairable with a vertex v in G, y $\in$ N(v). Now the subgraph induced by the vertices u, v, w and y is C_4. Hence the theorem. ■

Theorem 12 If a graph G is copairable, then CP(G) $\subseteq$ CP(S(G)) and CP(G) = CP(S(G)) if and only if Δ(G) = n-1 and δ(G) = 1.

Proof Let G be a copairable graph. Let V(G) = $\{v_1,v_2,\ldots,v_n\}$ and V(S(G)) = $\{v_1,v_2,\ldots,v_n;w_1,w_2,\ldots w_n\}$ such that $N(w_i)$ = V(G) $\cap N(v_i)$ for i = 1,2,…n. Let v_i and v_j be any two copairable vertices in G. Then $N(v_i) = N(v_j)^c$ in G. It is easy to note that $N(v_i) = N(v_j)^c$ in S(G). Therefore, CP(G) $\subseteq$ CP(S(G)). Let u,v $\in$ CP(S(G)). Since the newly added vertices in the splitting graph are independent, we have u and v both cannot be newly added vertices. We have already seen that if u,v $\in$ V(G), then u,v $\in$ CP(G). Without loss of generality, let u be a newly added vertex in S(G) and v be a vertex in G. We claim that v is a full vertex. If v is not a full vertex, then there exists a vertex $v_j \in$ V(G) such that $vv_j \notin$ E(G) and $vw_j \notin$ E(G). But uv $\in$ E(G) and hence u $\neq w_j$. But the newly added vertices are independent. Therefore, $w_j \notin$ N(u). Now N(u) $\cup N(v_j) \neq$ V(G). This is a contradiction. Hence v is a full vertex. It follows that u is a pendant vertex. Therefore, G must contain a pendant vertex so that S(G) contains a corresponding newly added vertex which is a pendant vertex. Hence CP(G) = CP(S(G)). And the converse is obvious. ■

Theorem 13 If a graph G is copairable, then CP(G) $\subseteq$ CP(CS(G)) and CS(G) is copairable if and only if v is an isolated vertex.

Proof Let G be a copairable graph. Let V(G) = $\{v_1,v_2,\ldots,v_n\}$ and V(CS(G)) = $\{v_1,v_2,\ldots,v_n;w_1,w_2,\ldots w_n\}$ such that $N(w_i) = N(v_i)^c \cap V(G)$ for i = 1,2,..,n. Let v_i and v_j be any two copairable vertices in G. Then $N(v_i) = N(v_j)^c$ in G. It is easy to note that $N(v_i) = N(v_j)^c$ in CS(G). Therefore, CP(G) $\subseteq$ CP(CS(G)). Let u,v $\in$ CP(CS(G)). Since the newly added vertices in the cosplitting graph are independent, then u and v both cannot be newly added vertices. We have already seen that if u and v are in V(G), then they are in CP(G). Let u' be a newly added vertex corresponding to some vertex u $\in$ V(G). We claim that v is an isolated vertex. Suppose not, then there exists a vertex w in CS(G) such that vw $\in$ E(G). Then w $\in$ N(u'). Otherwise, w $\in$ N(u) $\cap$ N(v) which is a contradiction. Now if the newly added vertex corresponding to w is w', then w' $\notin$ N(u) $\cup$ N(v) which is a contradiction. Hence v is an isolated vertex in G. And the converse is obvious. ■

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