Some Results on degree splitting graphs

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Abstract: **Let G(V, E) be a graph and let V_i denote the set of all vertices of degree i. The degree splitting graph DS(G) of G is obtained from G, by adding a new vertex w_i for each partition V_i such that $| V_i | \geq 2$ and joining w_i to each vertex of V_i. In this paper, we characterise graphs for which degree splitting graphs are regular, biregular bipartite.**

Keywords: **Splitting graph, co – splitting graph, degree splitting graph, k – regular adjacency vertex, k – regular adjacency graph.**

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1 Introduction

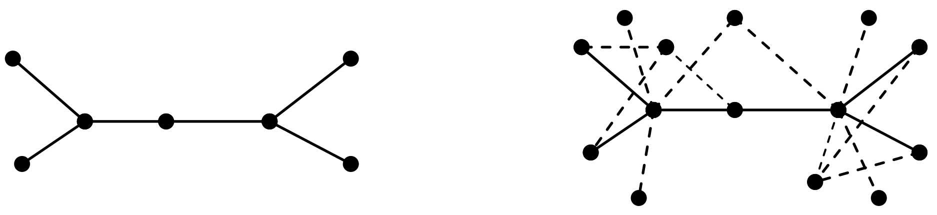
Only finite, simple, undirected graphs are considered in this paper. For notations and terminology, we follow [7]. The degree of a vertex v is denoted by d(v). Let V_i denote the set of all vertices of degree i. A graph G is said to be *r – regular*, if every vertex of G has degree r. For $r \neq k$, a graph G is said to be *(r,k) – biregular* if d(v) is either r or k for any vertex v in G. A *1 – factor* of G is a 1 – regular spanning subgraph of G and it is denoted by F.

For any vertex $v \in V(G)$, the *open neighbourhood* N(v) of v is the set of all vertices adjacent to v. That is, $N(v) = \{u \in V(G) / uv \in E(G)\}$. The *closed neighbourhood* of v is defined by $N[v] = N(v) \cup \{v\}$. The *join* of two graphs G_1 and G_2 is denoted by $G_1 \vee G_2$. In a graph G(V,E), let $V_i = \{v \in V / d(v) = i\}$. Note that in an r – regular graph G, $V_r = V(G)$ and $V_i = \phi$ for any $i \neq r$. A vertex v in G is called a *distinguished vertex* of G if v is the only vertex in G with degree d(v).

Analysing a graph by varying its vertex set and edge set is an interesting and well worked branch of research in graph theory. The studies on antipodal graphs and radial graphs[1,2,3,8,9] are examples in which the edge set of a graph varies itself. While studies based on the minimum number of vertices to be added to a graph[6] to attain a special property vary both of its vertex set and edge set. This branch of research has credited its account with one when Sampath Kumar and Walikar introduced the concept of splitting graph[11] of a graph. This concept resembles the method of taking clone of each vertex in a graph.

The definition of splitting graph is given as follows:

The graph S(G) obtained from G, by taking a new vertex v′ for every vertex $v \in V$ and joining v′ to all vertices of G adjacent to v, is called a s*plitting graph* of G. As an illustration, a graph G with its splitting graph S(G) is shown in Figure 1.



G **S(G)**

Figure 1

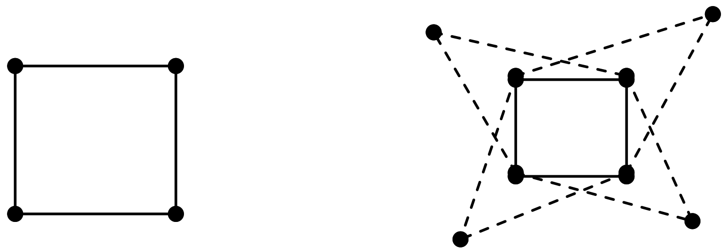
The properties of splitting graphs have been studied and the necessary and sufficient condition for a graph to be a splitting graph has been discussed in [11]. In fact, it has been proved that,

Result 1.1 [11] A graph G is a splitting graph if and only if V(G) can be partitioned into two sets V_1 and V_2 such that there exists a bijective mapping f from V_1 to V_2 and $N(f(v)) = N(v) \cap V_1$, for any $v \in V_1$.

On a similar line, recently instead of adding a clone for every vertex, a method of adding complement to every vertex has been introduced and studied for its properties[4].

Let G be a graph with vertex set $\{v_1, v_2, \ldots, v_n\}$. The *cosplitting graph* CS(G) is the graph obtained from G, by adding a new vertex w_i for each vertex v_i and joining w_i to all vertices which are not adjacent to v_i in G.

For instance, a graph G and its cosplitting graph CS(G) are shown in Figure 2.



G CS(G)

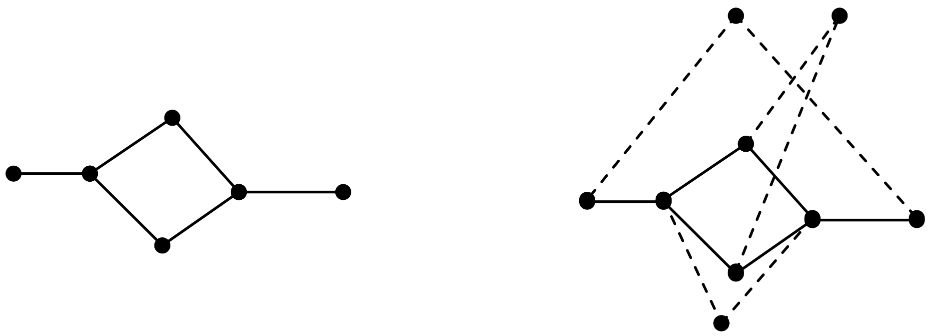
Figure 2

In [4], a necessary and sufficient condition for a graph to be a cosplitting graph has been given. And the graphs for which the splitting graph and the cosplitting graph are isomorphic have been characterised.

This branch spreads out its wings further with the introduction of degree splitting graph by Ponraj and Somasundaram [10].

The degree splitting graph DS(G) of a graph G can be defined as follows: For a graph G = (V, E), the *degree splitting graph* DS(G) is obtained from G, by adding a new vertex w_i for each partition V_i that contains at least two vertices and joining w_i to each vertex of V_i.

For example, a graph G and its degree splitting graph DS(G) are shown in Figure 3.



G DS(G)

Figure 3

It is easy to note that, if G is regular, then DS(G) is nothing but $G \vee K_1$ and trivial graph is the only graph for which the degree splitting graph is isomorphic to itself. Also every graph is an induced subgraph of its own degree splitting graph.

The definition of the degree splitting graph has inspired us very much as it resembles the process of assigning representatives for people having same qualities from the society of diverse qualities. It may also be compared to uniting systems according to their common nature in a network. Hence studying the properties of degree splitting graphs surely yield results of practical importance.

In this paper, we characterize the graphs for which degree splitting graph is regular. Also, we introduce the concept of RA graph here and characterize all graphs for which degree splitting graph is a biregular RA graph.

Further a study on distance in degree splitting graphs has been carried out in [5].

2 Characterization of degree splitting graphs

Let $d^*(v)$ denote the degree of a vertex v in DS(G) and d(v) denote that of v in G.

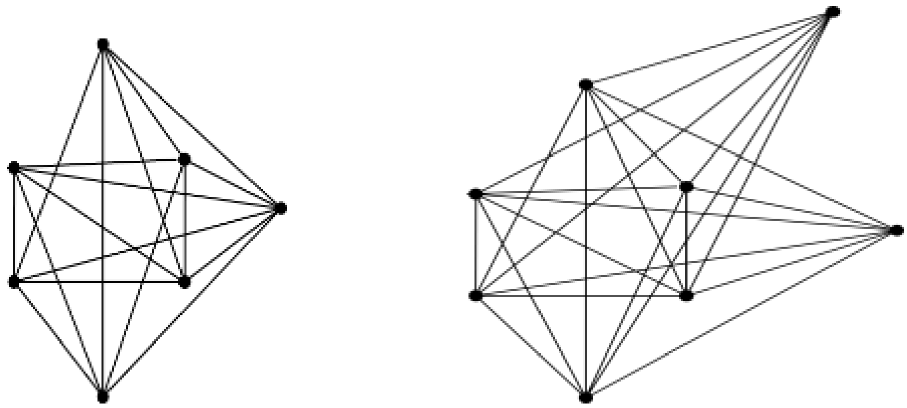
Theorem 2.1 The degree splitting graph DS(G) is regular if and only $G \cong K_r$ or $(K_{2k} - F) \vee K_1$, where F is a 1 – factor of K_{2k} and $k \geq 1$.

Proof Let G be any graph for which the degree splitting graph DS(G) is r – regular. Then $d^*(v)$ =

r, for all $v \in V(DS(G))$. Since for any vertex v of G, $d^*(v) = d(v)$ or $d(v) + 1$, $V(G) = V_r \cup V_{r-1}$. Here $|V_{r-1}| \geq 2$. Let w_{r-1} be the newly added vertex corresponding to V_{r-1} in DS(G). Since $d^*(w_{r-1}) = r$, we have $|V_{r-1}| = r$.

Now, $|V_r| \leq 1$, otherwise $V_{r+1} \neq \phi$ in V(DS(G)), which is impossible. If $|V_r| = 1$, then G contains r vertices of degree r – 1 and a single vertex of degree r. Thus r is even and so $G \cong (K_r - F) \vee K_1$. Otherwise, $|V_r| = 0$. Then G contains only r vertices of degree r – 1 and so $G \cong K_r$.

Conversely, if $G \cong K_r$, then $DS(G) \cong K_{r+1}$, which is r - regular and if $G \cong (K_{2k} - F) \vee K_1$, then DS(G) $\cong (K_{2k} - F) \vee K_2^c$, which is again a 2k - regular graph. For example, $(K_6 - F) \vee K_1$ with their degree splitting graphs are shown in Figure 4 ■



$(K_6\text{-}F)\vee K_1$ $\qquad$ $DS((K_6\text{-}F)\vee K_1)$

Figure 4

Since all the newly added vertices are adjacent to vertices of same degree, we need to refer this property by a common name so that the further reference becomes easier. For that we define RA vertices and RA graphs.

A non pendant vertex v is said to be a *k - regular adjacency vertex* (or simply a k - RA vertex) if d(u) = k for all $u \in N(v)$. A vertex is called an RA vertex if it is a k – RA vertex for some $k \geq 1$. For example, the newly added vertices w in DS(G) are all RA vertices.

A graph G in which every vertex is an RA vertex, is said to be an *RA graph.* For example, an RA graph is shown in Figure 5.

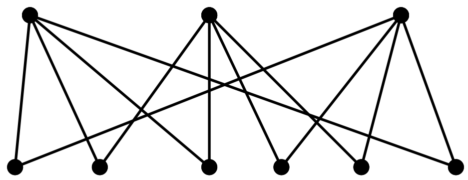


Figure 5

Obviously all regular graphs and $K_{m,n}$ are RA graphs.

As any graph G contains at least two vertices of same degree, any degree splitting graph DS(G) contains at least one RA vertex.

Now we prove two theorems which characterize all RA graphs.

Theorem 2.2 Any biregular RA graph is bipartite.

Proof Let G be an (r, k) – biregular RA graph. Then G contains only RA vertices. Take $X = \{v \in V(G) / d(v) = r\}$ and $Y = \{v \in V(G) / d(v) = k\}$. Clearly, $V(G) = X \cup Y$. We claim that (X, Y) is a bipartition of G. Let x and y be two vertices in Y and let z be in X. Assume that x and y are adjacent in G. Since G is connected, there is a (x, z) - path or a (y, z) - path of length at least two which contains an internal vertex w in Y. In such a case, N(w) includes a vertex in X and a vertex in Y, which is a contradiction to our assumption that G is an RA graph. Hence x and y cannot be adjacent in G. Similarly, no two vertices of X are adjacent in G. Thus G is bipartite. ■

Theorem 2.3 Any connected RA graph is either regular or biregular.

Proof Let G be a connected RA graph. If G is regular, then there is nothing to prove. Therefore, assume that G is not regular. Then there are two adjacent vertices x and y such that d(x) = a, d(y) = b and $a \neq b$. But every vertex in G is an RA vertex. Thus the vertices of every path that contains x are alternately of degree a and b. For every $z \in V(G)$, since there exists an (x, z) – path, d(z) is a or b and hence G is biregular. ■

Corollary 2.4 Any connected RA graph is either a regular graph or a biregular bipartite graph. ■

Notation Let $K_{n,2n+1}$ be the complete bipartite graph with bipartition (X,Y) where $X = \{v_1, v_2, \dots, v_n\}$ and $Y = \{w_1, w_2, \dots, w_{2n+1}\}$. Then $K^*_{n,2n+1}$ is the graph obtained from $K_{n,2n+1}$ by deleting the edges v_iw_{2i-1} and v_iw_{2i} for all i, $1 \leq i \leq n$. For example $K^*_{2,5}$ is shown in Figure 6.

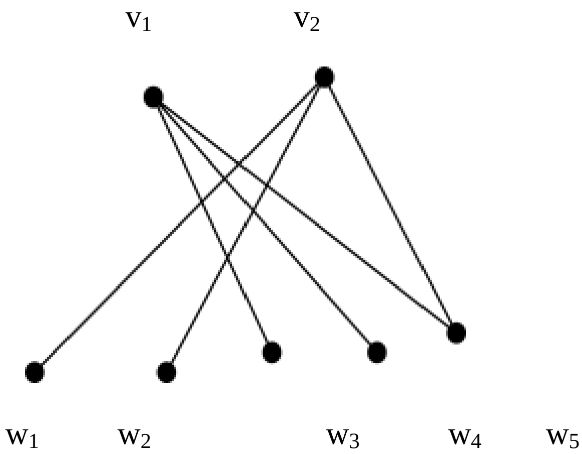


Figure 6

Theorem 2.5 Let G be a graph. Then DS(G) is a biregular RA graph if and only if $G \cong K_{1,n}$ or $K^*_{n,2n+1}$.

Proof Let G be any graph such that DS(G) is an (r,k) – biregular RA graph. Without loss of generality, assume that r > k. By Theorem 2.2, DS(G) is bipartite with bipartition (V^*_r, V^*_k), where $V^*_r = \{v \in V(DS(G)) / d(v) = r\}$ and $V^*_k = \{v \in V(DS(G)) / d(v) = k\}$.

Then, we have $V(G) = V_r \cup V_k \cup V_{r-1} \cup V_{k-1}$. Since G is a subgraph of DS(G), G is also bipartite with bipartition (X,Y) such that $X = V_r \cup V_{r-1}$ and $Y = V_k \cup V_{k-1}$.

If $|V_r| \geq 2$, then $d^*(v) = r + 1$, for all $v \in V_r$, which is a contradiction, since DS(G) is (r, k) biregular. Thus $|V_r| \leq 1$. Similarly we have $|V_k| \leq 1$.

If $|V_{r-1}| = 1$, then clearly $d^*(v) = r - 1$ for the vertex $v \in V_{r-1}$, which is a contradiction and so, $|V_{r-1}| \neq 1$. Similarly $|V_{k-1}| \neq 1$.

Since DS(G) is bipartite, the newly added vertex w_{r-1}, corresponding to V_{r-1} belongs to V^*_k in DS(G) and hence of degree k. Therefore, $|V_{r-1}| = k$. Similarly we have $|V_{k-1}| = r$.

When r – k = 1, $V(G) = V_r \cup V_{r-1} \cup V_{k-1}$ such that $X = V_r \cup V_{r-1}$ and $Y = V_{k-1}$ or $X = V_r$ and $Y = V_k \cup V_{k-1}$. The first case is invalid since r > k and the second case is valid only when r = k = 1 which is possible only when $G \cong K^*_{1,3}$ which is disconnected.

Now assume that $r - k \geq 2$. In order to analyze the possible structures of G, we discuss the following cases:

Case 1 Let $|V_r| = |V_k| = 0$.

Then $V = V_{r-1} \cup V_{k-1}$. If $V_{r-1} = \phi$ or $V_{k-1} = \phi$, then G is regular and so, $DS(G) = G \vee K_1$ which is not bipartite, a contradiction. On the other hand, if $|V_{r-1}| = k$ and $|V_{k-1}| = r$, then since G is bipartite, we have (r-1)k = (k-1)r, that is, k = r, which is again a contradiction.

Case 2 Let $| V_r | = | V_k | = 1$.

If $| V_{r-1} | = 0 = | V_{k-1} |$, then $G \cong K_2$ and so, $DS(G) \cong K_3$, a contradiction, since DS(G) is biregular. If $| V_{r-1} | = 0$ and $| V_{k-1}| \neq 0$, then we get $r = r(k-1) + k$. This forces that $k = \frac{2r}{r+1}$ and hence $r = k = 1$, a contradiction. Similarly the case when $| V_{k-1} | = 0$ and $| V_{r-1} | \neq 0$ is also impossible.

Let $| V_{r-1} | = k$ and $| V_{k-1} | = r$. Then we get $k(r-1) + r = r(k-1) + k$, that implies $r = k$, a contradiction.

Case 3 Let $| V_r | = 0$ and $| V_k | = 1$.

Then $| V_{r-1} |$ and $| V_{k-1} |$ cannot be both equal to zero. If $| V_{r-1} | = 0$ and $| V_{k-1}| \neq 0$, then since G is bipartite, we get a contradiction. If $| V_{k-1} | = 0$ and $| V_{r-1}| \neq 0$, then $k(r-1) = k$, which implies $r = 2$. Then $G \cong K_2$, which is a contradiction.

Let $| V_{r-1} | = k$ and $| V_{k-1} | = r$, therefore we have $k(r-1) = r(k-1) + k$ and so $r = 2k$. Therefore, G has $3k + 1$ vertices with k vertices of degree $2k - 1$in V_r^* and $2k + 1$ vertices in V_k^* in which 2k vertices are of degree $k – 1$ and a single vertex is of degree k, that is, $G \cong K^*_{k,2k+1}$ and hence $DS(G) \cong K^*_{k+1,2k+2}$.

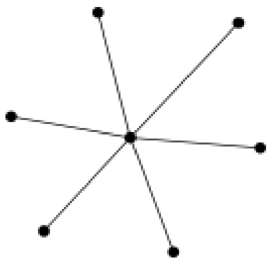
Case 4 Let $| V_k | = 0$ and $| V_r | = 1$.

V_{k-1} and V_{r-1} cannot be both empty sets. If $| V_{k-1} | = 0$ and $| V_{r-1}| \neq 0$, then since G is bipartite, we get a contradiction. If $| V_{r-1} | = 0$ and $| V_{k-1}| \neq 0$, then $V = V_r \cup V_{k-1}$. Then, we have $r = r(k – 1)$ which gives $k = 2$. Therefore, $G \cong K_{1,r}$ and $DS(G) \cong K_{2,r}$.

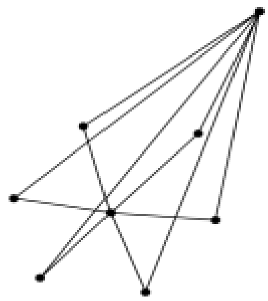
Let $| V_{r-1} | > 1$ and $| V_{k-1} | > 1$. Then $k(r-1) + r = r(k-1)$, that implies, $k = r$, a contradiction.

Thus, we can conclude that if DS(G) is a biregular RA graph, then $G\cong K_{1,r}$ or $K^*_{n,2n+1}$. The converse is obvious. For example, the graphs $K_{1,6}$ and $K^*_{2,5}$ with their degree splitting graphs are shown in Figure 7.

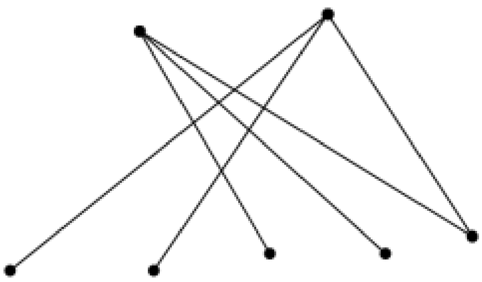
■



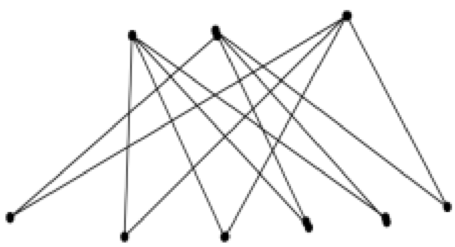
$K_{1,6}$



$DS(K_{1,6})$



$K^*_{2,5}$



$DS(K^*_{2,5})$

Figure 7

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