Zero –Free Regions of Analytic Functions with Restricted Coefficients

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Abstract: **In this paper we find bounds for the zeros of an analytic function by putting certain conditions on its coefficients.**

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1. Introduction

An elegant result in the theory of distribution of zeros of polynomials is the following theorem known as Enestrom-Kakeya Theorem[4]:

Theorem A: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n such that

$$a_n \geq a_{n-1} \geq ...... \geq a_1 \geq a_0 > 0.$$

Then all the zeros of P(z) lie in $|z| \leq 1$.

Aziz and Mohammad [1] extended the above theorem to a class of analytic functions $f(z)=\sum_{j=0}^{\infty}a_j z^j$ (not identically equal to zero) with its coefficients satisfying an Enestrom-Kakeya type condition. In fact, they proved the following result:

Theorem B: Let $f(z)=\sum_{j=0}^{\infty}a_j z^j$ (not identically equal to zero) be analytic in $|z| \leq t$. If $a_j > 0$ and $a_{j-1} - ta_j \geq 0,\ j = 1,2,3,......$, then f(z) does not vanish in $|z| < t$.

Aziz and Shah [2] gave a generalization of Theorem B and proved the following result:

Theorem C: Let $f(z)=\sum_{j=0}^{\infty}a_j z^j$ (not identically equal to zero) be analytic in $|z| \leq t$. If for some $k \geq 1$

$$ka_0 \geq ta_1 \geq t^2 a_2 \geq .... ,$$

then f(z) does not vanish in

$$\left| z - \left(\frac{k-1}{2k-1}\right)t \right| \leq \frac{kt}{2k-1}.$$

Shah and Liman [5] extended Theorem C to functions with complex coefficients and proved the following:

Theorem D: Let $f(z)=\sum_{j=0}^{\infty}a_j z^j$ (not identically equal to zero) be analytic in $|z| \leq t$. If for some $k \geq 1$

$$k|a_0| \geq t|a_1| \geq t^2|a_2| \geq .... ,$$

and for some real α, β,

$$|\arg a_j - \beta| \leq \alpha \leq \frac{\pi}{2},\ j = 0,1,2,...... ,$$

then f(z) does not vanish in

$$\left| z - \frac{(k-1)t}{M^2-(k-1)^2} \right| \leq \frac{Mt}{M^2-(k-1)^2},$$

where

$$M = k(\cos\alpha + \sin\alpha) + 2\frac{\sin\alpha}{|a_0|}\sum_{j=1}^{\infty}|a_j|t^j .$$

In this paper we prove the following results:

Theorem 1: Let $f(z)=\sum_{j=0}^{\infty}a_j z^j$ (not identically equal to zero) be analytic in $|z| \leq t$. If for some $k \leq 1$

$$k|a_0| \le t|a_1| \le t^2|a_2| \le ...., $$

and for some real α, β,

$$|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2},\ j = 0,1,2,......,$$

then f(z) does not vanish in

$$\left|z - \frac{(k-1)t}{M^2-(k-1)^2}\right| \le \frac{Mt}{M^2-(k-1)^2},$$

where

$$M = k(\sin\alpha - \cos\alpha) + 2\frac{\sin\alpha}{|a_0|}\sum_{j=1}^{\infty}|a_j|t^j .$$

Theorem 2: Let $f(z) = \sum_{j=0}^{\infty} a_j z^j$ (not identically equal to zero) be analytic in $|z| \le t$. If for some $k \le 1$,

$$k|a_0| \le t|a_1| \le t^2|a_2| \le .... \le t^{\lambda}|a_{\lambda}|$$
$$\ge t^{\lambda+1}|a_{\lambda+1}| \ge ......,$$

and for some real α, β,

$$|\arg a_j - \beta| \le \alpha \le \frac{\pi}{2},\ j = 0,1,2,......,$$

then f(z) does not vanish in

$$\left|z - \frac{(k-1)t}{M'^2-(k-1)^2}\right| \le \frac{M't}{M'^2-(k-1)^2},$$

where

$$M' = (2t^{\lambda}\left|\frac{a_\lambda}{a_0}\right| - k)\cos\alpha + k\sin\alpha + 2\frac{\sin\alpha}{|a_0|}\sum_{j=1}^{\infty}|a_j|t^j .$$

2. Lemma

For the proofs of the above results , we need the following lemma due to Govil and Rahman [3]:

Lemma: If b_1, b_2 are real numbers such that $b_1 \ge b_2$ and for some real α, β,

$$|b_j - \beta| \le \alpha \le \frac{\pi}{2},\ j = 1,2 .$$

Then

$$|b_1 - b_2| \le (|b_1| - |b_2|)\cos\alpha + (|b_1| + |b_2|)\sin\alpha .$$

3. Proofs of Theorems

Proof of Theorem1: Since $f(z) = \sum_{j=0}^{\infty} a_j z^j$ is analytic in $|z| \le t$, we have $\lim_{j\to\infty} a_j z^j = 0$.

Now consider the function

$$F(z) = (t-z)f(z) = (t-z)(a_0 + a_1 z + a_2 z^2 + ...)$$

$$= ta_0 + (ta_1 - a_0)z + (ta_2 - a_1)z^2 + ...$$
$$= ta_0 - a_0 z + ka_0 z - (ka_0 - ta_1)z + \sum_{j=2}^{\infty}(ta_j - a_{j-1})t^j$$
$$= ta_0 - a_0 z + ka_0 z + G(z)$$

where

$$G(z) = -(ka_0 - ta_1)z + \sum_{j=2}^{\infty}(ta_j - a_{j-1})z^j .$$

For $|z| = t$, we have by using the hypothesis and the above lemma,

$$|G(z)| \le |ka_0 - ta_1|t + \sum_{j=2}^{\infty}|ta_j - a_{j-1}|t^j$$

$$\le t(t|a_1| - k|a_0|)\cos\alpha + t(t|a_1| + k|a_0|)\sin\alpha$$

$$+ t^2\{(t|a_2| - |a_1|)\cos\alpha + (t|a_2| + |a_1|)\sin\alpha$$
$$+ t^3\{(t|a_3| - |a_2|)\cos\alpha + (t|a_3| + |a_2|)\sin\alpha\} + ...$$
$$+ t^{\lambda}\{(t|a_\lambda| - |a_{\lambda-1}|)\cos\alpha + (t|a_\lambda| - |a_{\lambda-1}|)\sin\alpha\}$$
$$+ t^{\lambda+1}\{(t|a_{\lambda+1}| - |a_\lambda|)\cos\alpha + (t|a_{\lambda+1}| - |a_\lambda|)\sin\alpha\} + ...$$

$$= t|a_0|[k(\sin\alpha - \cos\alpha) + 2\frac{\sin\alpha}{|a_0|}\sum_{j=1}^{\infty}|a_j|t^j]$$

$$= t|a_0|M$$

Since G(z) is analytic for $|z| \le t$,G(0)=0, it follows by Schwarz lemma that

$$|G(z)| \le t|a_0|M|z| \quad \text{for} \quad |z| \le t .$$

Hence , for $|z| \le t$

$$\begin{aligned} |F(z)| &= |ta_0 - a_0 z + ka_0 z + G(z)| \\ &\ge |ta_0 - a_0 z + ka_0 z| - |G(z)| \\ &\ge |a_0|[|(k-1)z + t| - tM|z|] \\ &> 0 \end{aligned}$$

if

$$tM|z| < |(k-1)z + t| .$$

It can be easily verified that the above region is precisely the disk

$$\left| z - \frac{(k-1)t}{M^2 - (k-1)^2} \right| < \frac{Mt}{M^2 - (k-1)^2} .$$

Thus , it follows that F(z) and hence f(z) does not vanish in

$$\left| z - \frac{(k-1)t}{M^2 - (k-1)^2} \right| < \frac{Mt}{M^2 - (k-1)^2}$$

and the proof of the theorem is complete.

Proof of Theorem 2: As in the proof of theorem 1, we have for $|z| \le t$,

$$|G(z)| \le t(t|a_1| - k|a_0|)\cos\alpha + t(t|a_1| + k|a_0|)\sin\alpha$$

$$+ t^2\{(t|a_2| - |a_1|)\cos\alpha + (t|a_2| + |a_1|)\sin\alpha$$
$$+ t^3\{(t|a_3| - |a_2|)\cos\alpha + (t|a_3| + |a_2|)\sin\alpha\} + ...$$
$$+ t^\lambda\{(t|a_\lambda| - |a_{\lambda-1}|)\cos\alpha + (t|a_\lambda| - |a_{\lambda-1}|)\sin\alpha\}$$
$$+ t^{\lambda+1}\{(|a_\lambda| - t|a_{\lambda+1}|)\cos\alpha + (|a_\lambda| - t|a_{\lambda+1}|)\sin\alpha\} + ...$$

$$= t|a_0|[2\frac{t^\lambda}{|a_0|} - k)\cos\alpha + k\sin\alpha + 2\frac{\sin\alpha}{|a_0|}\sum_{j=1}^{\infty}|a_j|t^j]$$
$$= t|a_0|M'.$$

Hence , as in the proof of Theorem 1, it follows that F(z) and hence f(z) does not vanish in

$$\left| z - \frac{(k-1)t}{M'^2 - (k-1)^2} \right| < \frac{M't}{M'^2 - (k-1)^2}$$

and the proof of theorem 2 is complete.

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