On The Regional Location of Zeros of a Polynomial

M.H.Gulzar [1], **B.A.Zargar** [2], **M.A.Mir** [3]
Department of Mathematics, University of Kashmir, India
gulzarmh@gmail.com
bazargar@gmail.com
mabdullah_mir@yahoo.co.in

Abstract: **The purpose of this paper is to present some extensions and generalizations of recently proved results in connection with the Enestrom-Kakeya Theorem.**

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1. Introduction

An elegant result in the theory of distribution of zeros of polynomials is the following known as Enestrom-Kakeya Theorem [7]
(see also[3,4,6,7,9]):

Theorem A: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n such that

$$a_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0 > 0.$$

Then all the zeros of P(z) lie in $|z| \le 1$.

As a generalization of Theorem A, Aziz and Zargar[1,2] proved the following results:

Theorem B: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n such that for some $k \ge 1$,

$$ka_n \ge a_{n-1} \ge ...... \ge a_1 \ge a_0.$$

Then all the zeros of P(z) lie in

$$|z+k-1| \le \frac{ka_n + |a_0| - a_0}{|a_n|}.$$

Theorem C: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n. If for some $\rho, 0<\rho\le 1$ and $\lambda, 0 \le \lambda \le n-1$,

$$a_n \le a_{n-1} \le ...... \le a_{\lambda+1} \le a_\lambda \ge a_{\lambda-1} \ge ...... \ge a_1 \ge \rho a_0,$$

Then all the zeros of P(z) lie in

$$\left|z+\frac{a_{n-1}}{a_n}-1\right| \le \frac{2a_\lambda - a_{n-1} + (2-\rho)|a_0| - \rho a_0}{|a_n|}.$$

Recently A.A.Mogbademu et al. [8] proved the following generalization of Theorem C:

Theorem D: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n with complex coefficients. If $\mathrm{Re}(a_j)=\alpha_j, \mathrm{Im}(a_j)=\beta_j, j=0,1,2,......, n$ and for some t>0; $\mu \ge 0; 0 \le \lambda \le n-1;$ $0<\rho\le 1,$,

$$t^n\alpha_n + \mu t^{n-1} \le t^{n-1}\alpha_{n-1} \le ...... \le t^{\lambda+1}\alpha_{\lambda+1} \le t^\lambda \alpha_\lambda \ge t^{\lambda-1}\alpha_{\lambda-1} \ge ...... \ge t\alpha_1 \ge \rho\alpha_0,$$

then all the zeros of P(z) lie in

$$\left|z-\frac{\mu}{a_n}\right| \le \frac{1}{|a_n|}\left[\frac{2\alpha_\lambda}{t^{n-\lambda-1}} + \mu - \alpha_n t + \sum_{j=1}^{n}\frac{|\beta_j t - \beta_{j-1}|}{t^{n-j}} + \frac{2|\alpha_0| - \rho(|\alpha_0| + \alpha_0) + |\beta_0|)}{t^{n-1}}\right.$$

2. Main Results

In this paper we present the following extension of Theorem D which also generalizes a result due to Govil and Mctume[5].

Theorem 1: Let $P(z)=\sum_{j=0}^{n}a_j z^j$ be a polynomial of degree n with complex coefficients. If $\mathrm{Re}(a_j)=\alpha_j, \mathrm{Im}(a_j)=\beta_j, j=0,1,2,......, n$ and for some t>0, $\rho_1 \ge 0; \rho_2 \ge 0;$ $0 \le \lambda \le n-1; 0 \le \mu \le n-1$ $;0<\sigma_1 \le 1$ and $\sigma_2; 0<\sigma_2 \le 1,$

$t^n\alpha_n + \rho_1 t^{n-1} \le t^{n-1}\alpha_{n-1} \le ...... \le t^{\lambda+1}\alpha_{\lambda+1}$
$\le t^{\lambda}\alpha_\lambda \ge t^{\lambda-1}\alpha_{\lambda-1} \ge ...... \ge t\alpha_1 \ge \sigma_1\alpha_0,$
$t^n\beta_n + \rho_2 t^{n-1} \le t^{n-1}\beta_{n-1} \le ...... \le t^{\lambda+1}\beta_{\lambda+1}$
$\le t^{\lambda}\beta_\lambda \ge t^{\lambda-1}\beta_{\lambda-1} \ge ...... \ge t\beta_1 \ge \sigma_2\beta_0,$

then all the zeros of P(z) lie in

$$\left|z - \frac{(\rho_1 + i\rho_2)}{a_n}\right| \le \frac{1}{|a_n|}\Big[\frac{2\alpha_\lambda}{t^{n-\lambda-1}} + \frac{2\beta_\mu}{t^{n-\mu-1}} + \rho_1 + \rho_2 - (\alpha_n + \beta_n)t + \frac{2(|\alpha_0| + |\beta_0|)}{t^{n-1}} - \sigma_1\frac{(\alpha_0 + |\alpha_0|)}{t^{n-1}} - \sigma_2\frac{(\beta_0 + |\beta_0|)}{t^{n-1}} \Big].$$

If a_j is real i.e. $\beta_j = 0, \forall j = 0,1,......, n$, then taking $\rho_1 = \rho, \rho_2 = 0, \sigma_1 = \sigma$, we get the following result from Theorem 1:

Corollary 1: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. If

$\mathrm{Re}(a_j) = \alpha_j, \mathrm{Im}(a_j) = \beta_j, j = 0,1,2,......, n$

and for some t>0; $\rho \ge 0; 0 \le \lambda \le n-1;$

$0 < \sigma \le 1,$

$$t^n a_n + \rho t^{n-1} \le t^{n-1}a_{n-1} \le ...... \le t^{\lambda+1}a_{\lambda+1} \le t^{\lambda}a_\lambda \ge t^{\lambda-1}a_{\lambda-1} \ge ...... \ge ta_1 \ge \sigma a_0,$$

then all the zeros of P(z) lie in

$$\left|z - \frac{\rho}{a_n}\right| \le \frac{1}{|a_n|}\Big[\frac{2a_\lambda}{t^{n-\lambda-1}} + \rho - a_n t + 2\frac{|a_0|}{t^{n-1}} - \sigma\frac{(a_0 + |a_0|)}{t^{n-1}} \Big].$$

Taking $\sigma = 1$ in Cor.1, we get the following result:

Corollary 2: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. If

$\mathrm{Re}(a_j) = \alpha_j, \mathrm{Im}(a_j) = \beta_j, j = 0,1,2,......, n$

and for some t>0; $\rho \ge 0; 0 \le \lambda \le n-1,$

$t^n a_n + \rho t^{n-1} \le t^{n-1}a_{n-1} \le ...... \le t^{\lambda+1}a_{\lambda+1} \le t^{\lambda}a_\lambda$
$\ge t^{\lambda-1}a_{\lambda-1} \ge ...... \ge ta_1 \ge a_0,$

then all the zeros of P(z) lie in

$$\left|z - \frac{\rho}{a_n}\right| \le \frac{1}{|a_n|}\Big[\frac{2a_\lambda}{t^{n-\lambda-1}} + \rho - a_n t + \frac{|a_0|}{t^{n-1}} - \frac{a_0}{t^{n-1}} \Big].$$

Taking $\sigma_1 = \sigma_2 = 1$ in Theorem 1, we get the following result: **Corollary 3:** Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. If

$\mathrm{Re}(a_j) = \alpha_j, \mathrm{Im}(a_j) = \beta_j, j = 0,1,2,......, n$

and for some t>0, $\rho_1 \ge 0, \rho_2 \ge 0,$

$0 \le \lambda \le n-1, 0 \le \mu \le n-1,$

$t^n\alpha_n + \rho_1 t^{n-1} \le t^{n-1}\alpha_{n-1} \le ...... \le t^{\lambda+1}\alpha_{\lambda+1}$
$\le t^{\lambda}\alpha_\lambda \ge t^{\lambda-1}\alpha_{\lambda-1} \ge ...... \ge t\alpha_1 \ge \alpha_0,$
$t^n\beta_n + \rho_2 t^{n-1} \le t^{n-1}\beta_{n-1} \le ...... \le t^{\mu+1}\beta_{\mu+1}$
$\le t^{\mu}\beta_\mu \ge t^{\mu-1}\beta_{\mu-1} \ge ...... \ge t\beta_1 \ge \beta_0,$

then all the zeros of P(z) lie in

$$\left|z - \frac{(\rho_1 + i\rho_2)}{a_n}\right| \le \frac{1}{|a_n|}\Big[\frac{2\alpha_\lambda}{t^{n-\lambda-1}} + 2\frac{\beta_\mu}{t^{n-\mu-1}} + \rho_1 + \rho_2 - (\alpha_n + \beta_n)t + \frac{|\alpha_0| + |\beta_0|)}{t^{n-1}} - \frac{(\alpha_0 + \beta_0)}{t^{n-1}} \Big].$$

Taking $\rho_1 = \rho_2 = 0$ in Theorem 1, we get the following result:

Corollary 4: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. If

$\mathrm{Re}(a_j) = \alpha_j, \mathrm{Im}(a_j) = \beta_j, j = 0,1,2,......, n$

and for some t>0,

$0 \le \lambda \le n-1, 0 \le \mu \le n-1, 0 < \sigma_1 \le 1$ and

$0 < \sigma_2 \le 1,$

$t^n\alpha_n \le t^{n-1}\alpha_{n-1} \le ...... \le t^{\lambda+1}\alpha_{\lambda+1} \le t^{\lambda}\alpha_\lambda \ge t^{\lambda-1}\alpha_{\lambda-1}$
$\ge ...... \ge t\alpha_1 \ge \sigma_1\alpha_0,$
$t^n\beta_n \le t^{n-1}\beta_{n-1} \le ...... \le t^{\mu+1}\beta_{\mu+1} \le t^{\mu}\beta_\mu \ge t^{\mu-1}\beta_{\mu-1}$
$\ge ...... \ge t\beta_1 \ge \sigma_2\beta_0,$

then all the zeros of P(z) lie in

$$|z| \le \frac{1}{|a_n|}\Big[\ \frac{2\alpha_\lambda}{t^{n-\lambda-1}} + 2\frac{\beta_\mu}{t^{n-\mu-1}} - (\alpha_n + \beta_n)t$$
$$+ 2\frac{(|\alpha_0| + |\beta_0|)}{t^{n-1}} - \sigma_1\frac{(\alpha_0 + |\alpha_0|)}{t^{n-1}} - \sigma_2\frac{(\beta_0 + |\beta_0|)}{t^{n-1}}\ \Big].$$

Taking t=1 in Theorem 1, we get the following result:

Corollary 5: Let $P(z) = \sum_{j=0}^{n} a_j z^j$ be a polynomial of degree n with complex coefficients. If $\operatorname{Re}(a_j) = \alpha_j, \operatorname{Im}(a_j) = \beta_j, j = 0,1,2,......, n$ and for some $\rho_1 \ge 0$, $\rho_2 \ge 0, \lambda, \mu; 0 \le \lambda \le n-1, 0 \le \mu \le n-1$, $\sigma_1; 0 < \sigma_1 \le 1$ and $\sigma_2; 0 < \sigma_2 \le 1$,

$$\alpha_n + \rho_1 \le \alpha_{n-1} \le ...... \le \alpha_{\lambda+1} \le \alpha_\lambda \ge \alpha_{\lambda-1} \ge ......$$
$$\ge \alpha_1 \ge \sigma_1\alpha_0,$$
$$\beta_n + \rho_2 \le \beta_{n-1} \le ...... \le \beta_{\mu+1} \le \beta_\mu \ge \beta_{\mu-1} \ge ......$$
$$\ge \beta_1 \ge \sigma_2\beta_0,$$

then all the zeros of P(z) lie in

$$\left| z - \frac{(\rho_1 + i\rho_2)}{a_n} \right| \le \frac{1}{|a_n|}\Big[\ 2\alpha_\lambda + 2\beta_\mu + \rho_1 + \rho_2$$
$$-(\alpha_n + \beta_n) + 2(|\alpha_0| + |\beta_0|) - \sigma_1(\alpha_0 + |\alpha_0|)$$
$$-\sigma_2(\beta_0 + |\beta_0|)\ \Big].$$

In a similar way we get many other interesting results by taking different values of the parameters in the above results.

3. Proofs of Theorems

Proof of Theorem 1: Consider the polynomial

$$F(z) = (t - z)P(z)$$
$$= (t - z)(a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0)$$
$$= -a_n z^{n+1} + \sum_{j=1}^{n}(a_j t - a_{j-1})z^j + a_0 t$$
$$= -a_n z^{n+1} + \sum_{j=1}^{n}(\alpha_j t - \alpha_{j-1})z^j + \alpha_0 t$$
$$+ i\{\sum_{j=1}^{n}(\beta_j t - \beta_{j-1})z^j + \beta_0 t\}$$
$$= -a_n z^{n+1} + (\alpha_n t - \alpha_{n-1})z^n + \sum_{j=1}^{n-1}(\alpha_j t - \alpha_{j-1})z^j$$
$$+ \alpha_0 t + i\{(\beta_n t - \beta_{n-1})z^n$$
$$+ \sum_{j=1}^{n-1}(\beta_j t - \beta_{j-1})z^j + \beta_0 t\}$$

$$= -a_n z^{n+1} + (\rho_1 - \alpha_n t)z^n + \alpha_n t z^n$$
$$+ (\alpha_n t - \rho_1 - \alpha_{n-1})z^n + (\alpha_{n-1} t - \alpha_{n-2})z^{n-1}$$

$$+ ...... + (\alpha_{\lambda+1} t - \alpha_\lambda)z^{\lambda+1} + (\alpha_\lambda t - \alpha_{\lambda-1})z^\lambda$$
$$+ ...... + (\alpha_1 t - \sigma\alpha_0)z + (\sigma_1\alpha_0 - \alpha_0)z + \alpha_0 t$$
$$+ i\{(\rho_2 - \beta_n t)z^n + \beta_n t z^n + (\beta_n t - \rho_2 - \beta_{n-1})z^n$$
$$+ (\beta_{n-1} t - \beta_{n-2})z^{n-1} + ... + (\beta_{\mu+1} t - \beta_\mu)z^{\mu+1}$$
$$+ (\beta_\mu t - \beta_{\mu-1})z^\mu + ... + (\beta_1 t - \sigma_2\beta_0)z$$
$$+ (\sigma_2\beta_0 - \beta_0)z + \beta_0 t\}.$$

Therefore ,for $|z| \ge t$ so that $\frac{1}{|z|^{n-j}} \le \frac{1}{t^{n-j}}$ for $0 \le j \le n$, we have

$$|F(z)| = | -a_n z^{n+1} + (\rho_1 - \alpha_n t) z^n + \alpha_n t z^n$$
$$+ (\alpha_n t - \rho_1 - \alpha_{n-1}) z^n + (\alpha_{n-1} t - \alpha_{n-2}) z^{n-1}$$
$$+ \ldots\ldots + (\alpha_{\lambda+1} t - \alpha_\lambda) z^{\lambda+1} + (\alpha_\lambda t - \alpha_{\lambda-1}) z^\lambda + .$$
$$\ldots.. + (\alpha_1 t - \sigma\alpha_0) z + (\sigma_1 \alpha_0 - \alpha_0) z + \alpha_0 t$$
$$+ i\{(\rho_2 - \beta_n t) z^n + \beta_n t z^n + (\beta_n t - \rho_2 - \beta_{n-1}) z^n$$
$$+ (\beta_{n-1} t - \beta_{n-2}) z^{n-1} + \ldots\ldots + (\beta_{\mu+1} t - \beta_\mu) z^{\mu+1}$$
$$+ (\beta_\mu t - \beta_{\mu-1}) z^\mu + \ldots\ldots + (\beta_1 t - \sigma_2 \beta_0) z$$
$$+ (\sigma_2 \beta_0 - \beta_0) z + \beta_0 t\} |$$

$$\geq |z|^n |a_n z - (\rho_1 + i\rho_2)| - |z|^n [\ |\alpha_n t - \rho_1 - \alpha_{n-1}|$$
$$+ \frac{|\alpha_{n-1} t - \alpha_{n-2}|}{|z|} + \ldots\ldots + \frac{|\alpha_{\lambda+1} t - \alpha_\lambda|}{|z|^{n-\lambda-1}} + \frac{|\alpha_\lambda t - \alpha_{\lambda-1}|}{|z|^{n-\lambda}}$$
$$+ \ldots\ldots + \frac{|\alpha_1 t - \sigma_1 \alpha_0|}{|z|^{n-1}} + \frac{|\sigma_1 \alpha_0 - \alpha_0|}{|z|^{n-1}} + \frac{|\alpha_0| t}{|z|^n}$$
$$+ |\beta_n t - \rho_2 - \beta_{n-1}| + \frac{|\beta_{n-1} t - \beta_{n-2}|}{|z|} + \ldots\ldots$$
$$+ \frac{|\beta_{\mu+1} t - \beta_\mu|}{|z|^{n-\mu-1}} + \frac{|\beta_\mu t - \beta_{\mu-1}|}{|z|^{n-\mu}}$$
$$+ \ldots\ldots + \frac{|\beta_1 t - \sigma_2 \beta_0|}{|z|^{n-1}} + \frac{|\sigma_2 \beta_0 - \beta_0|}{|z|^{n-1}} + \frac{|\beta_0| t}{|z|^n}\ ]$$

$$\geq |z|^n [\ |a_n z - (\rho_1 + i\rho_2)| - \{|\alpha_n t - \rho_1 - \alpha_{n-1}|$$
$$+ \frac{|\alpha_{n-1} t - \alpha_{n-2}|}{t} + \ldots\ldots + \frac{|\alpha_{\lambda+1} t - \alpha_\lambda|}{t^{n-\lambda-1}} + \frac{|\alpha_\lambda t - \alpha_{\lambda-1}|}{t^{n-\lambda}}$$
$$+ \ldots\ldots + \frac{|\alpha_1 t - \sigma_1 \alpha_0|}{t^{n-1}} + \frac{|\sigma_1 \alpha_0 - \alpha_0|}{t^{n-1}} + \frac{|\alpha_0| t}{t^n}$$
$$+ |\beta_n t - \rho_2 - \beta_{n-1}| + \frac{|\beta_{n-1} t - \beta_{n-2}|}{t} + \ldots\ldots$$
$$+ \frac{|\beta_{\mu+1} t - \beta_\mu|}{t^{n-\mu-1}} + \frac{|\beta_\mu t - \beta_{\mu-1}|}{t^{n-\mu}}$$
$$+ \ldots\ldots + \frac{|\beta_1 t - \sigma_2 \beta_0|}{t^{n-1}} + \frac{|\sigma_2 \beta_0 - \beta_0|}{t^{n-1}} + \frac{|\beta_0| t}{t^n}\}\ ]$$

$$\geq |z|^n [\ |a_n z - (\rho_1 + i\rho_2)| - \{-\alpha_n t + \rho_1$$
$$+ \alpha_{n-1} - \alpha_{n-1} + \frac{\alpha_{n-2}}{t} + \ldots\ldots$$
$$- \frac{\alpha_{\lambda+1}}{t^{n-\lambda-2}} + \frac{\alpha_\lambda}{t^{n-\lambda-1}} + \frac{\alpha_\lambda}{t^{n-\lambda-1}} - \frac{\alpha_{\lambda-1}}{t^{n-\lambda}} + \ldots\ldots$$
$$+ \frac{\alpha_1}{t^{n-2}} - \frac{\sigma_1 \alpha_0}{t^{n-1}} - \frac{\sigma_1 |\alpha_0|}{t^{n-1}} + 2\frac{|\alpha_0|}{t^{n-1}}$$
$$- \beta_n t + \rho_2 + \beta_{n-1} - \beta_{n-1} + \frac{\beta_{n-2}}{t} + \ldots\ldots$$
$$- \frac{\beta_{\mu+1}}{t^{n-\mu-2}} + \frac{\beta_\mu}{t^{n-\mu-1}} + \frac{\beta_\mu}{t^{n-\mu-1}} - \frac{\beta_{\mu-1}}{t^{n-\mu}}$$
$$+ \ldots\ldots + \frac{\beta_1}{t^{n-2}} - \frac{\sigma_2 \beta_0}{t^{n-1}} - \frac{\sigma_2 |\beta_0|}{t^{n-1}} + 2\frac{|\beta_0|}{t^{n-1}}\}\ ]$$

$$= |z|^n [\ |a_n z - (\rho_1 + i\rho_2)| - \{\frac{2\alpha_\lambda}{t^{n-\lambda-1}} + \rho_1 - \alpha_n t$$
$$+ 2\frac{|\alpha_0|}{t^{n-1}} - \sigma_1 \frac{(\alpha_0 + |\alpha_0|)}{t^{n-1}} + 2\frac{\beta_\mu}{t^{n-\mu-1}} + \rho_2$$
$$- \beta_n t + 2\frac{|\beta_0|}{t^{n-1}} - \sigma_2 \frac{(\beta_0 + |\beta_0|)}{t^{n-1}}\}\ ]$$
$$> 0$$

if

$$|a_n z - (\rho_1 + i\rho_2)| > \frac{2\alpha_\lambda}{t^{n-\lambda-1}} + 2\frac{\beta_\mu}{t^{n-\mu-1}} + \rho_1 + \rho_2$$
$$- (\alpha_n + \beta_n) t + 2\frac{(|\alpha_0| + |\beta_0|)}{t^{n-1}} - \sigma_1 \frac{(\alpha_0 + |\alpha_0|)}{t^{n-1}}$$
$$- \sigma_2 \frac{(\beta_0 + |\beta_0|)}{t^{n-1}}.$$

This shows that those zeros of F(z) whose modulus is greater than or equal to t lie in

$$\left|z-\frac{(\rho_1+i\rho_2)}{a_n}\right|\le\frac{1}{|a_n|}\Big[\ \frac{2\alpha_\lambda}{t^{n-\lambda-1}}+2\frac{\beta_\mu}{t^{n-\mu-1}}+\rho_1+\rho_2$$
$$-(\alpha_n+\beta_n)t+2\frac{(|\alpha_0|+|\beta_0|)}{t^{n-1}}-\sigma_1\frac{(\alpha_0+|\alpha_0|)}{t^{n-1}}$$
$$-\sigma_2\frac{(\beta_0+|\beta_0|)}{t^{n-1}}\ \Big].$$

Since the zeros of F(z) whose modulus is less than t already satisfy the above inequality and since all the zeros of F(z) are also the zeros of P(z), it follows that all the zeros of P(z) lie in

$$\left|z-\frac{(\rho_1+i\rho_2)}{a_n}\right|\le\frac{1}{|a_n|}\Big[\ \frac{2\alpha_\lambda}{t^{n-\lambda-1}}+2\frac{\beta_\mu}{t^{n-\mu-1}}+\rho_1+\rho_2$$
$$-(\alpha_n+\beta_n)t+2\frac{(|\alpha_0|+|\beta_0|)}{t^{n-1}}-\sigma_1\frac{(\alpha_0+|\alpha_0|)}{t^{n-1}}$$
$$-\sigma_2\frac{(\beta_0+|\beta_0|)}{t^{n-1}}\ \Big].$$

That completes the proof of Theorem 1.

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