

# Analysis of Stress Conditions on Composite Layered Pressure Vessel with Solid works Simulation

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**Abstract** – The aim is to study of the various design optimization techniques by using finite element analysis on solid works simulation for a composite layered pressure vessel with internal and external stress conditions and find out what thickness of plate is required and what are the number of plates required with certain composition and material so that strength can be created which is able to handle the pressure created by the by the gases from inside, this will explain how by changing different layer of different material will result in stresses formed at different cross section of a boiler. We will also be able to see that if reduce the number of plates and change the sequence and what effect will it have on the strength after shell.

**Keywords** – Pressure vessel, Solidworks 2014 with simulation license, References of composite layered shells.

## I. INTRODUCTION

Pressure vessels are used in many industries (e.g., hydrocarbon processing, chemical, power, pharmaceutical, food and beverage). The mechanical design of most pressure vessels is done in accordance with the requirements contained in the ASME Boiler and Pressure Vessel Code, Section VIII. Section VIII is divided into three divisions. [1][2] This course provides an overview of pressure vessel mechanical design requirements. It focuses on highlights of the differences in scope among the several factors related to pressure vessel design that the ASME Code does not cover. The following summarizes the main sections of the course collected in the winter time contains some useful information.

### a) General

- 1.Main Pressure Vessel Components
- 2.Primary Process Functions of Pressure Vessels
- 3.Scope of ASME Code Section VIII
- 4.Structure of Section VIII, Division 1

### b) Materials of Construction

- 1.Material Selection Factors
- 2.Maximum Allowable Stress

### c) Design

- 1.Design Conditions and Loadings
- 2.Weld Joint Efficiency and Corrosion Allowance
- 3.Design for Internal Pressure

- 4.Design for External Pressure and Compressive Stresses
- 5.Reinforcement of Openings
- 6.Flange Rating
- 7.Flange Design
- 8.Maximum Allowable Working Pressure

### d) Other Design Considerations

- 1.Vessel Support
- 2.Local Loads
- 3.Vessel Internals

### e) Fabrication

- 1.Acceptable Welding Details
- 2.Post weld Heat Treatment Requirements

### f) Inspection and Testing

- 1.Inspection
- 2.Pressure Testing



Fig. – 1.1 Simulation View of Pressure vessel

This course is nominally only ½ day in length. Therefore, it cannot provide an in dept treatment of all aspects of pressure vessel design. However, the topics are covered in sufficient

depth to provide participants with a general understanding of pressure vessel design requirements, to design pressure vessel components to a limited extent, or to review pressure vessel designs prepared by others. It also prepares individuals who require a more thorough understanding of pressure vessel design to attend a more in-depth course or to acquire the necessary knowledge on their own.

#### g) General

This section describes the various components of pressure vessels through the use of conceptual drawings. It also describes the scope of the ASME Boiler and Pressure Vessel Code Section VIII, and the basic structure of Section VIII, Division 1.

#### h) Main Pressure Vessel Components

Pressure vessels are containers for fluids that are under pressure. They are used in a wide variety of industries (e.g., petroleum refining, chemical, power, pulp and paper, food, etc.).

##### 1. Shell

The shell is the primary component that contains the pressure. Pressure vessel shells are welded together to form a structure that has a common rotational axis. Most pressure vessel shells are cylindrical, spherical, or conical in shape.

##### 2. Head

All pressure vessel shells must be closed at the ends by heads (or another shell section). Heads are typically curved rather than flat. Curved configurations are stronger and allow the heads to be thinner, lighter, and less expensive than flat heads. Figures 2.1 through 2.4 show heads closing the cylindrical sections of the subject pressure vessels. Heads can also be used inside a vessel. These "intermediate heads" separate sections of the pressure.

#### i). Materials of Construction

This section discusses the primary factors that influence material selection for pressure vessels and the maximum allowable material stresses specified by the ASME Code. The mechanical design of a pressure vessel can proceed only after the materials have been specified. The ASME Code does not state what materials must be used in each application. It specifies what materials may be used for ASME Code vessels, plus rules and limitations on their use. But, it is up to the end user to specify the appropriate materials for each application considering various material selection factors in conjunction with ASME Code requirements

##### 1.1. Material Selection Factors

The main factors that influence material selection are:

- a) Strength
- b) Corrosion Resistance

- c) Resistance to Hydrogen Attack
- d) Fracture Toughness

#### 1. Strength

Strength is a material's ability to withstand an imposed force or stress. Strength is a significant factor in the material selection for a particular application. Strength determines how thick a component must be to withstand the imposed loads. The overall strength of a material is determined by its yield strength, ultimate tensile strength, creep and rupture strengths. These strength properties depend on the chemical composition of the material. Creep resistance (a measure of material strength at elevated temperature) is increased by the addition of alloying elements such as chromium, molybdenum, and/or nickel to carbon steel. Therefore, alloy materials are often used in elevated temperature applications.

#### 2. Resistance to Hydrogen Attack

At temperatures from approximately 300°F to 400°F, monatomic hydrogen diffuses into voids that are normally present in steel. In these voids, the monatomic hydrogen forms molecular hydrogen, which cannot diffuse out of the steel. If this hydrogen diffusion continues, pressure can build to high levels within the steel. At elevated temperatures, over approximately 600°F, monatomic hydrogen not only causes cracks to form but also attacks the steel. Hydrogen attack differs from corrosion in that damage occurs throughout the thickness of the component, rather than just at its surface, and occurs without any metal loss. In addition, once hydrogen attack has occurred, the metal cannot be repaired and must be replaced.

#### 3. Fracture Toughness

Fracture toughness refers to the ability of a material to withstand conditions that could cause a brittle fracture. The fracture toughness of a material can be determined by the magnitude of the impact energy that is required to fracture a specimen using Charpy V-notch test.

## 2. EXPERIMENT

### Design

#### 1. Design Conditions and Loadings

The mechanical design of a pressure vessel begins with specification of the design pressure and design temperature. Pressure imposes loads that must be withstood by the individual vessel components. Temperature affects material strength and, thus, its allowable stress, regardless of the design pressure. Some pressure vessels have multiple sets of design conditions that correspond to different modes of operation. For example, during its operating cycle, a reactor may have a high pressure and moderate temperature during normal operation, but it may operate at a much lower pressure and a very high temperature during catalyst regeneration. Both sets of design conditions must be specified because either one or

the other may govern the mechanical design.



Fig 1.2 static displacement (pressure vessel) in simulation

All pressure vessels must be designed for the most severe conditions of coincident pressure and temperature that are expected during normal service. Normal service includes conditions that are associated with:

- Startup.
- Normal operation.
- Deviations from normal operation that can be anticipated.
- Shutdown.

Pressure vessels must also be designed for other loading conditions and service factors that may apply in particular situations. These are highlighted later.

Pressure

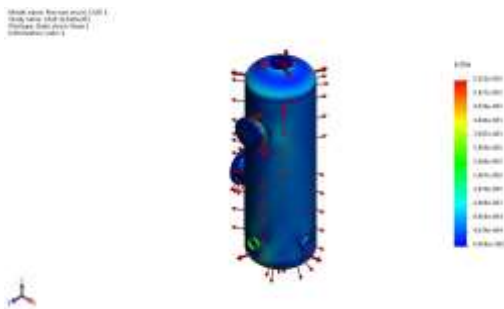


Fig. 1.3 static strain (pressure vessel) in simulation

### Operating Pressure

The operating pressure must be set based on the maximum internal or external pressure that the pressure vessel may encounter. The following factors must be considered:

- Ambient temperature effects.
  - Normal operational variations.
  - Pressure variations due to changes in the vapor pressure of the contained fluid.
  - Pump or compressor shut-off pressure.
  - Static head due to the liquid level in the vessel.
- System pressure drop.

- Normal pre-startup activities or other operating conditions that may occur (e.g., vacuum), that should be considered in the design.
- Design Pressure
- Generally, design pressure is the maximum internal pressure that is used in the mechanical design of a pressure vessel. For full or partial vacuum conditions, the design
- Pressure is applied externally and is the maximum pressure difference that can occur between the atmosphere and the inside of the pressure vessel. Some pressure vessels may
- Experience both internal and external pressure conditions at different times during their operation. The mechanical design of the pressure vessel in this case is based on which of these is the more severe design condition. The specified design pressure is based on the maximum operating pressure at the top of the vessel, plus the margin that the process design engineer determines is suitable for the particular application. A suitable margin must also be provided between the maximum operating pressure and the safety relief valve set pressure. This margin is necessary to prevent frequent and unnecessary opening of the safety relief valve that may occur during normal variations in operating pressure. The safety relief valve set pressure is normally set equal to the design pressure.

## 2.1 METHODOLOGY

- Selection of type of design of shell.
- Modeling of design
- Attributing material
- Inducing design into Solidworks simulation express.
- Comparison of various iteration of results.

## 2.2 OBJECTIVE

The aim is to find out what thickness of plate is required and what are the number of plates required with certain composition and material so that strength can be created which is able to handle the pressure created by the by the gases from inside, this will explain how by changing different layer of different material will result in stresses formed at different cross section of a boiler. We will also be able to see that if reduce the number of plates and change the sequence and what effect will it have on the strength after shell

## 2.3 - Solidworks Simulation -










Model name: CASE 2 FINAL Pressure vessel\_CASE 1

Current Configuration: Default

## Solid Bodies



| <L_MdInf_SldBd_Nm/>   | Treated As | Volumetric Properties   | Document Modified  | Path/Date            |
|---|------------|---|--|----------------------|
| Thicken2<br> | Solid Body | Mass:217.037 kg<br>Volume:0.027609 m <sup>3</sup><br>Density:7861.1 kg/m <sup>3</sup><br>Weight:2126.97 N   | E:\CASE 2 FINAL\CASE 2 FINALHead - full.SLDPRT           | Nov 05 08:17:34 2015 |
| Thicken1  | Solid Body | Mass:226.721 kg<br>Volume:0.0288409 m <sup>3</sup><br>Density:7861.1 kg/m <sup>3</sup><br>Weight:2221.87 N  | E:\CASE 2 FINAL\CASE 2 FINALHead - full.SLDPRT           | Nov 05 08:17:34 2015 |
| Thicken1  | Solid Body | Mass:219.02 kg<br>Volume:0.0278613 m <sup>3</sup><br>Density:7861.1 kg/m <sup>3</sup><br>Weight:2146.4 N    | E:\CASE 2 FINAL\CASE 2 FINALHead.SLDPRT                  | Nov 05 08:12:46 2015 |
| Thicken2  | Solid Body | Mass:209.478 kg<br>Volume:0.0266474 m <sup>3</sup><br>Density:7861.1 kg/m <sup>3</sup><br>Weight:2052.88 N  | E:\CASE 2 FINAL\CASE 2 FINALHead.SLDPRT                  | Nov 05 08:12:46 2015 |
| CirPattern1   | Solid Body | Mass:18.5617 kg<br>Volume:0.00236121 m <sup>3</sup><br>Density:7861.1 kg/m <sup>3</sup><br>Weight:181.905 N | E:\CASE 2 FINAL\CASE 2 FINALSteam outlet flange.SLDPRT   | Oct 22 14:21:36 2013 |
| Cut-Extrude2  | Solid Body | Mass:84.3741 kg<br>Volume:0.0107331 m <sup>3</sup><br>Density:7861.1 kg/m <sup>3</sup><br>Weight:826.866 N  | E:\CASE 2 FINAL\CASE 2 FINALManhole cover.sldprt         | Oct 22 14:21:36 2013 |
| CirPattern1   | Solid Body | Mass:38.7128 kg<br>Volume:0.0049246 m <sup>3</sup><br>Density:7861.1 kg/m <sup>3</sup><br>Weight:379.385 N  | E:\CASE 2 FINAL\CASE 2 FINALManhole nozzle flange.sldprt | Oct 22 14:21:36 2013 |

|  |             |   |   |
|--|-------------|---|---|
| Thicken2<br>      | Solid Body  | Mass:715.139 kg<br>Volume:0.0909719 m <sup>3</sup><br>Density:7861.1 kg/m <sup>3</sup><br>Weight:7008.36 N                    | E:\CASE 2 FINAL\CASE 2<br>FINALShell 1.sldprt<br>Nov 05 08:15:24 2015                   |
| Thicken1<br>      | Solid Body  | Mass:728.508 kg<br>Volume:0.0926726 m <sup>3</sup><br>Density:7861.1 kg/m <sup>3</sup><br>Weight:7139.38 N                    | E:\CASE 2 FINAL\CASE 2<br>FINALShell 1.sldprt<br>Nov 05 08:15:24 2015                   |
| Nozzle 1<br>      | Solid Body  | Mass:912.49 kg<br>Volume:0.116077 m <sup>3</sup><br>Density:7861.1 kg/m <sup>3</sup><br>Weight:8942.4 N                       | E:\CASE 2 FINAL\CASE 2<br>FINALShell 2.sldprt<br>Nov 05 08:16:30 2015                   |
| Thicken1<br>      | Solid Body  | Mass:895.83 kg<br>Volume:0.113957 m <sup>3</sup><br>Density:7861.1 kg/m <sup>3</sup><br>Weight:8779.14 N                      | E:\CASE 2 FINAL\CASE 2<br>FINALShell 2.sldprt<br>Nov 05 08:16:30 2015                   |
| Thicken2<br>    | Solid Body  | Mass:879.399 kg<br>Volume:0.111867 m <sup>3</sup><br>Density:7861.1 kg/m <sup>3</sup><br>Weight:8618.11 N                     | E:\CASE 2 FINAL\CASE 2<br>FINALShell 2.sldprt<br>Nov 05 08:16:30 2015                   |
| CirPattern1<br> | Solid Body  | Mass:94.3722 kg<br>Volume:0.012005 m <sup>3</sup><br>Density:7861.1 kg/m <sup>3</sup><br>Weight:924.847 N                     | E:\CASE 2 FINAL\CASE 2<br>FINALSteam inlet<br>flange.sldprt<br>Oct 22 14:21:36 2013     |
| Shell Bodies   |             |   |   |
| <L_MdInf_ShIBd_Nm/>  | Formulation | Volumetric Properties   | Document Path/Date<br>Modified  |
| Shell-2<br>     | Thin        | Thickness:12.7 mm<br>Weight:2291.73 N<br>Volume:0.0297478 m <sup>3</sup><br>Mass:233.85 kg<br>Density:7861.1kg/m <sup>3</sup> | E:\CASE 2 FINAL\CASE 2<br>FINALHead.SLDPR<br>Nov 05 08:12:46 2015                       |
| Shell-3  | Thin        | Thickness:4.7625 mm<br>Weight:0 N<br>Volume:0 m <sup>3</sup><br>Mass:0 kg<br>Density:-1.#INDkg/m <sup>3</sup>                 | E:\CASE 2 FINAL\CASE 2<br>FINALPressure vessel_CASE<br>1.SLDASM<br>Nov 05 08:19:35 2015 |

|          |      |  |  |
|----------|------|--|--|
| Shell-4  | Thin | Thickness:4.7625 mm<br>Weight:28.3961 N<br>Volume:0.000368595 m <sup>3</sup><br>Mass:2.89756 kg<br>Density:7861.1kg/m <sup>3</sup> | E:\CASE 2 FINAL\CASE 2<br>FINALSteam outlet<br>nozzle.SLDPRT<br>Oct 22 14:21:36 2013 |
| Shell-5  | Thin | Thickness:6.35 mm<br>Weight:26.3993 N<br>Volume:0.000342675 m <sup>3</sup><br>Mass:2.6938 kg<br>Density:7861.1kg/m <sup>3</sup>    | E:\CASE 2 FINAL\CASE 2<br>FINALLug nozzle.sldprt<br>Nov 04 22:46:07 2015             |
| Shell-6  | Thin | Thickness:9.525 mm<br>Weight:84.3845 N<br>Volume:0.00109535 m <sup>3</sup><br>Mass:8.61067 kg<br>Density:7861.1kg/m <sup>3</sup>   | E:\CASE 2 FINAL\CASE 2<br>FINALLug<br>reinforcement.sldprt<br>Nov 04 22:46:07 2015   |
| Shell-7  | Thin | Thickness:6.35 mm<br>Weight:26.3993 N<br>Volume:0.000342675 m <sup>3</sup><br>Mass:2.6938 kg<br>Density:7861.1kg/m <sup>3</sup>    | E:\CASE 2 FINAL\CASE 2<br>FINALLug nozzle.sldprt<br>Nov 04 22:46:07 2015             |
| Shell-8  | Thin | Thickness:9.525 mm<br>Weight:84.3845 N<br>Volume:0.00109535 m <sup>3</sup><br>Mass:8.61067 kg<br>Density:7861.1kg/m <sup>3</sup>   | E:\CASE 2 FINAL\CASE 2<br>FINALLug<br>reinforcement.sldprt<br>Nov 04 22:46:07 2015   |
| Shell-9  | Thin | Thickness:6.35 mm<br>Weight:26.3993 N<br>Volume:0.000342675 m <sup>3</sup><br>Mass:2.6938 kg<br>Density:7861.1kg/m <sup>3</sup>    | E:\CASE 2 FINAL\CASE 2<br>FINALLug nozzle.sldprt<br>Nov 04 22:46:07 2015             |
| Shell-10 | Thin | Thickness:9.525 mm<br>Weight:84.3845 N<br>Volume:0.00109535 m <sup>3</sup><br>Mass:8.61067 kg<br>Density:7861.1kg/m <sup>3</sup>   | E:\CASE 2 FINAL\CASE 2<br>FINALLug<br>reinforcement.sldprt<br>Nov 04 22:46:07 2015   |
| Shell-11 | Thin | Thickness:6.35 mm<br>Weight:26.3993 N<br>Volume:0.000342675 m <sup>3</sup><br>Mass:2.6938 kg<br>Density:7861.1kg/m <sup>3</sup>    | E:\CASE 2 FINAL\CASE 2<br>FINALLug nozzle.sldprt<br>Nov 04 22:46:07 2015             |
| Shell-12 | Thin | Thickness:9.525 mm<br>Weight:84.3845 N<br>Volume:0.00109535 m <sup>3</sup><br>Mass:8.61067 kg<br>Density:7861.1kg/m <sup>3</sup>   | E:\CASE 2 FINAL\CASE 2<br>FINALLug<br>reinforcement.sldprt<br>Nov 04 22:46:07 2015   |

|  |       |  |  |
|--|-------|--|--|
| Shell-13<br>              | Thin  | Thickness:12.7 mm<br>Weight:135.261 N<br>Volume:0.00175575 m <sup>3</sup><br>Mass:13.8021 kg<br>Density:7861.1kg/m <sup>3</sup>  | E:\CASE 2 FINAL\CASE 2 FINALManhole reinforcement .SLDPRT<br>Nov 04 22:46:08 2015    |
| Shell-14<br>              | Thin  | Thickness:4.7625 mm<br>Weight:132.021 N<br>Volume:0.0017137 m <sup>3</sup><br>Mass:13.4715 kg<br>Density:7861.1kg/m <sup>3</sup> | E:\CASE 2 FINAL\CASE 2 FINALManhole nozzle.sldprt<br>Nov 04 22:46:07 2015            |
| Shell-15<br>              | Thin  | Thickness:12.7 mm<br>Weight:7339.91 N<br>Volume:0.0952756 m <sup>3</sup><br>Mass:748.97 kg<br>Density:7861.1kg/m <sup>3</sup>    | E:\CASE 2 FINAL\CASE 2 FINALShell 1.sldprt<br>Nov 05 08:15:24 2015                   |
| Shell-16<br>              | Thick | Thickness:4.7625 mm<br>Weight:0 N<br>Volume:0 m <sup>3</sup><br>Mass:0 kg<br>Density:-1.#INDkg/m <sup>3</sup>                    | E:\CASE 2 FINAL\CASE 2 FINALPressure vessel_CASE 1.SLDASM<br>Nov 05 08:19:35 2015    |
| Shell-17<br>            | Thin  | Thickness:12.7 mm<br>Weight:228.545 N<br>Volume:0.00296662 m <sup>3</sup><br>Mass:23.3209 kg<br>Density:7861.1kg/m <sup>3</sup>  | E:\CASE 2 FINAL\CASE 2 FINALSteam inlet reinforcement.sldprt<br>Nov 04 22:46:08 2015 |
| Shell-18<br>            | Thin  | Thickness:25.4 mm<br>Weight:1125.15 N<br>Volume:0.014605 m <sup>3</sup><br>Mass:114.811 kg<br>Density:7861.1kg/m <sup>3</sup>    | E:\CASE 2 FINAL\CASE 2 FINALSteam inlet nozzle.sldprt<br>Oct 22 14:21:36 2013        |
| Shell-21<br>            | Thin  | Thickness:12.7 mm<br>Weight:2368.05 N<br>Volume:0.0307385 m <sup>3</sup><br>Mass:241.638 kg<br>Density:7861.1kg/m <sup>3</sup>   | E:\CASE 2 FINAL\CASE 2 FINALHead - full.SLDPRT<br>Nov 05 08:17:34 2015               |
| Copy[ 20 ] Shell-21<br> | Thin  | Thickness:1 mm<br>Weight:567.316 N<br>Volume:0.00736404 m <sup>3</sup><br>Mass:57.8894 kg<br>Density:7861.1kg/m <sup>3</sup>     | E:\CASE 2 FINAL\CASE 2 FINALShell 1.sldprt<br>Nov 05 08:15:24 2015                   |



|   |      |   |  |
|---|------|---|--|
| Shell-22<br> | Thin | Thickness:1 mm<br>Weight:173.72 N<br>Volume:0.00225497 m <sup>3</sup><br>Mass:17.7265 kg<br>Density:7861.1kg/m <sup>3</sup>     | E:\CASE 2 FINAL\CASE 2<br>FINALHead.SLDPRT<br>Nov 05 08:12:46 2015                   |
| Shell-23<br> | Thin | Thickness:1 mm<br>Weight:6.16397 N<br>Volume:8.00113e-005 m <sup>3</sup><br>Mass:0.628976 kg<br>Density:7861.1kg/m <sup>3</sup> | E:\CASE 2 FINAL\CASE 2<br>FINALSteam outlet<br>nozzle.SLDPRT<br>Oct 22 14:21:36 2013 |

**Table - 1.1 Simulation analysis of internal and external pressure on solid vessel**

### 3. CONCLUSION

The research in the field of pressure and its effects with respect to stresses has been widely experimented with various methods, however the composite layered method has been limitedly used as of few reasons material technology, now because of the latest simulation techniques of the software's it's easy to visualize.

That's why the use of this method has been widely felt relevance. And further researches in this field have been implemented therefore for it has been observed that if this method is applied on boilers with given constraints that will increase the accuracy of result as well as open new dimensions which can make engineering more specific. Comparison seen in the research work will help to explain the stresses and situations at point of different time on different areas of the boiler this will also explain how the same study can be carried out to various other mechanical components which is of the same kind of analysis this study will also cover up the limitations with conventional methods and also increased degree of accuracy hence finding out the stresses or the local stresses where the mesh created can be made finer of course and because of this analysis will become more responsible the significance of the proposed research covers how the plate thickness or the composition of the material of the shell can be changed it different options or iterations so as to achieve desired result composite.

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