Optimal Solutions of Mean-Field Games Models & Applications

G.sasikala[#1], Dr.S.Kumara ghuru[#2]
Department of mathematics, Chikkanna Govt Arts college, Tirupur, Tamilnadu, India
mutharasan2k@gmail.com
kumarunmath@yahoo.co .in

Abstract - **Mean – field Games are studied by means of reference case applications. In this article, we provide reference case in two main reasons. One is simple for explicit resolution, based on Bellman functions are Quadratic and Stationary measures. Secondly, inspire of its simplicity, the case is rich enough in terms of more complex models that may not be as tractable as this one. The stability can be deal with explicitly using Hermit Polynomials.**

Key words - **Mean Field Games, Reference case, Partial Differential case, Control Theory, Eigen Vectors.**

I. INTRODUCTION

The approach of Mean Field Games(MFG) have been initiated by Lasry and Lions(2006) has provided an elegant and tractable way to study approximate Nash – equilibra for large population, stochastic differential games with a so called Mean Field interaction. In such games, the players private state processes are coupled only through their empirical distributions.

1.1 MFG in Game Theory

A set of strategies for the finite players are derived from the solution of this limiting problem. These strategies form an approximate Nash equilibrium for the n-player game if 'n' is large .In this sense that no player can improve his expected reward by more than C_n by unilaterally changing his strategy where $C_n \rightarrow 0$ as $n \rightarrow \infty$. They are attractive in that they are distribution the strategy of a single player depends only on his own private state. Intuitively if n is large, because of the symmetry of the model, players i's contribution to μ^n is negligible and he may as well treat μ^n as fixed.

The finite player games studied in the article that the dynamics of player – private state process are given by a stochastic differential Game as

$$d\, x_i^i = \beta\left(t, x^i, \mu^n, \alpha_i^i\right) dt + \lambda\,(t, x^i)\, dw_i^i$$

$$x_0^i = \gamma^i \quad \text{......................... (1)}$$

Where μ^n the empirical distribution of the states

$$\mu^n = \frac{1}{n}\sum_{j=1}^{n} \delta x^j \quad \text{.......................... (2)}$$

The drift β may be depend on time, player i's private state **ε**, the distribution of the private states and the player i's own choice of control λ_i^i and w_i^i are independent winner process ,again $\gamma^i$are independent identically distributed random variables independent of the winner processes and each player has the same drift and volatility coefficients. Moreover each player i have to same objective.

The objective is to minimize as,

$$E\left[\int_0^T F\left(t, x^i, \mu^n, q_i^n, \alpha_i^i\right) dt + G\,(x^i, \mu^n)\right]$$

Where $q_i^n = \frac{1}{n}\sum_{j=1}^{n} \delta \alpha_i^i$

Over all admissible choices of α_i^i ,

Subject to the constraint

$$d\, x_i^i = \beta\left(t, x^i, \mu^n, \alpha_i^i\right) dt + \lambda\,(t, x^i)\, dw_i^i$$

$$x_0^i = \gamma^i$$

1.2 MFG in Economics

The heart of modern economics that is the general equilibrium theory can be considered a mean field game (MFG) where the mean field is obviously the vector of prices. Prices are indeed a relevant summary of interactions between agents and in turn, they influence each agent behavior. This approach certainly clarifies what a market exists because of agents interactions and in turn, the market induces individual behaviors. For a particular suitable example of Mean Field of Games which are in fact a general tool to embed externality in models since mean fields are not constrained to be prices. Penetration rates for technologies such as wind turbines or solar panels are instances of Mean Field.

II. BASIC DEFINITIONS

2.1 Frame Work of MFG

The General Frame Work of MFG theory is consists of the following four classes. They are (i) Agents anonymity (ii) Continuum of Agents (iii) Rational Expectations (iv) Social interaction of the Mean Field Type.

2.2 Agents Anonymity

Agent anonymity has always been implicit in game theory but is worth recalling method. Basically, it says that agents are anonymous in the sense that any permutation of the agents does not change the outcome of the game.

2.3 Continuum of Agents

Continuum Agents is often used to model games with large number of players. It is rather well accepted approximation that has been used to tractability purposes and here, for the Mean Field Games, the limit of a game with N-players as $N \rightarrow \infty$ has been studied in J.M.Lassy and P.L.Lions.

2.4 Rational Expectations

Rational Expectations has been introduced in the 60's and now widely accepted among game theory researchers.

2.5 Social Interactions of the Mean Field Approach

This is specific to Mean Field Games and is a hypothesis on interactions between players. The main idea is that a given agent cannot take into account every single agent is going to interact with.(ie) Every agent is going to make a decision according to some statistics regarding the overall community of agents.

III. FRAME WORK APPLICATIONS

Consider a continuum of individuals (or population) that has preferences about resembling each other. This type of problem is typically of the mean field game start where individuals pay a price to move from one point to another in the state space and have a utility flow that is a function of the overall distribution of individuals in the population.

3.1 Mathematical Formulation of Mean Field Games PDE

Let us consider a special case the system of coupled PDE's by a single PDE of Hamiltonian – Jacobi equation which is given by

$$\frac{\sigma^2}{2}\Delta u + H(\nabla u) - \rho u = -g(m) \qquad \ldots\ldots\ldots\ldots(3)$$

and Kolmogorov equation which is given by

$$\partial_t m + \Delta u\left(mH'(\nabla u)\right) = \frac{\sigma^2}{2}\Delta^m \qquad \ldots\ldots\ldots\ldots(4)$$

Where $H(\nabla u) = \underset{a}{max}\ [a\nabla u - h(a)]$ and 'm' is the probability distribution function.

Our aim is to find the optional solution as stationary of the problem in several special cases replacing T by ∞ .(ie) T→∞

(ie) The modified PD Equations are given by Hamilton Jacob

$$\frac{\sigma^2}{2}\Delta u + H(\nabla u) - \rho u = -g(m) \qquad \ldots\ldots\ldots\ldots(5)$$

&Kolmogoro $is\ \left(mH'(\nabla u)\right) = \frac{\sigma^2}{2}\Delta u$ …………………(6)

Where m is the probability distribution function.

3.2 Frame work under Quadratic costs

The quadratic costs Frame Work is formulated by a simple Harmonic equation

$[H(P) = \frac{1}{2}P^2]$

with quadratic costs, the system can be written as follows Hamilton – Jacobi is

$$\partial_t u + \frac{\sigma^2}{2}\Delta u + \frac{1}{2}[\nabla u]^2 - \rho u = -g(m)$$

$$\partial_t u + \frac{\sigma^2}{2}\Delta u + H(\nabla u) = -g(m)$$

This implies

$$\partial_t u + \frac{\sigma^2}{2}\Delta u + \frac{1}{2}[\nabla u]^2 - \rho u = -g(m) \qquad \ldots\ldots\ldots(7)$$

& Kolmogorov equation is

$$\partial_t m + \nabla\left(m(\nabla u)\right) = \frac{\sigma^2}{2}\Delta m \qquad \ldots\ldots\ldots(8)$$

The equations (7), (8) in its stationary form the system is reduced to simple.

(ie) (7) is $\frac{\sigma^2}{2}\Delta u + \frac{1}{2}[\nabla u]^2 - \rho u = -g(m)$ ……..………(9)

(8) is $\nabla.\left(m(\nabla u)\right) = \frac{\sigma^2}{2}\Delta m$ ……………(10)

Next, we consider the problem exactly to $\Psi = \sqrt{m}$ as one partial differential equation.

3.3 Proposition I: Statement.

With quadratic cost, the system can be written as

$$\partial_1 u + \frac{\sigma^2}{2}\Delta u + \frac{1}{2}[\nabla u]^2 - \rho u = -g(m)$$

(Hamilton jacobi)

$$\partial_1 m + \nabla\left(m(\nabla u)\right) = \frac{\sigma^2}{2}\Delta m$$

(kolmogrov)

The system is simply when its form, is given by

$$\frac{\sigma^2}{2}\Delta u + \frac{1}{2}[\nabla u]^2 - \rho u = -g(m)$$

$$\nabla.\left(m(\nabla u)\right) = \frac{\sigma^2}{2}\Delta m$$

3.4 Solution procedure for proposition I.

The Kolmogorov equation

$$\frac{\sigma^2}{2}\Delta u(x) + \frac{1}{2}[\nabla u(x)]^2 - \rho u(x) = -g\left[e^{\frac{2u(x)}{\sigma^2}}\right] \qquad \ldots\ldots(11)$$

Taking log on both sides,

$$\log \alpha + \log\ \Delta u(x) + \frac{1}{2}log\ [\nabla u(x)^2 - \rho u(x)] = \log\frac{1}{g} + \frac{2u(x)}{\sigma^2}$$

Differentiating,

$$\frac{1}{\Delta u(x)}D\Delta u(x) + \frac{1}{2}\ \frac{[2\nabla u(x) - \rho Du(x)]}{\nabla u(x)]2 - \rho u(x)} = \frac{2u(x)}{\sigma^2} \qquad \ldots\ldots(12)$$

We get $\nabla m = \frac{2\Delta u)}{\sigma^2}\ m$

Applying divergence on both sides

$$\nabla.(\nabla m) = \nabla\left[\frac{2\Delta u)}{\sigma^2}m\right]$$

$$\Longrightarrow \quad \nabla.(m\nabla u) = \frac{\sigma^2}{2}\Delta m \qquad \ldots\ldots\ldots(13)$$

Which is kolmogrov equation.

3.5 Proposition II: Statement.

Let us consider a couple(K, Ψ) Where K is a scalar.If (K, Ψ) is a solution of

$$\frac{\sigma^4}{2}\ \frac{\nabla\Psi(x)}{\Psi(x)} = \rho\sigma^2\ \log\left[\frac{\Psi(x)}{k}\right] - g\ [\Psi^2(x)] \qquad \ldots\ldots\ldots(14)$$

$$\int[\Psi(x)]^2\,dx = 1$$

Then $m = \Psi^2$ and $u = \sigma^2\ \log\left[\frac{\Psi}{k}\right]$ are solutions of our inintial stationary problem.

3.6 Solution procedure for proportion II

Consider (K, Ψ) is solution of the above equation (14), introduce

$\Psi = \sqrt{m}$ Implies $m = \Psi^2$ ...………….(15)

And

u = σ^2 log $\left[\frac{\Psi}{k}\right]$ = σ^2 [log (Ψ) - log k] …………..(16)

⟹ u = σ^2 [log (Ψ) - log k]

Taking derivative, then we have,

$\frac{\nabla m}{m} = \alpha \frac{\nabla \Psi}{\Psi}$ ……………..(17)

$\nabla u = \sigma^2 \frac{\nabla \Psi}{\Psi} - 0$

$\nabla u = \frac{\sigma^2}{2} \nabla m$ (using (17) ..……………(18)

Hence (u,m) variables of the Hamilton – Jacobi equation.

IV.FRAME WORK UNDER EIGEN VALUES PROBLEM

4.1 Definition : Hermite polynomials

The Hermite polynomials of n^{th} order of L^2[m(x)]dx is defined by

$H_n(x) = S^n \frac{1}{n!} (-1)^n e^{\left(\frac{x^2}{2s^2}\right)} \frac{d^n}{dx^n}\left[e^{\left(-\frac{x^2}{2s^2}\right)}\right]$ ………………. (19)

If n = 1, $H_1(x) = - S \frac{1}{1!} e^{\left(\frac{x^2}{2s^2}\right)} \frac{d}{dx}\left[e^{\left(-\frac{x^2}{2s^2}\right)}\right]$

If n = 2 , $H_2(x) = S^2 \frac{1}{2!} (-1)^2 e^{\left(\frac{x^2}{2s^2}\right)} \frac{d^2}{dx^2}\left[e^{\left(-\frac{x^2}{2s^2}\right)}\right]$

4.2 Frame work considers Eigen values.

Proposition III: The Hermit polynomials H_n are Eigen vectors of $\mathcal{L}$ and $\mathcal{L} H_n = 2\,\mathcal{E}\,n$

H_n where $\mathcal{L}$ is Hilbert space $\mathcal{L}^2$[m*(x)dx] .

Solution procedure:

Let [A_n] be square matrix defined by

$A_n = \begin{bmatrix} \rho + 2\,\mathcal{E}\,n & -1 \\ \frac{n}{s^2} & -2\,\mathcal{E}\,n \end{bmatrix}$

Now we choose n≥2.The Eigen values of A_n are of opposite signs, $\lambda_n^1 < 0 < \lambda_n^2$

With $\lambda_n^{1,2} = \frac{1}{2} [\rho \pm \sqrt{(\rho^2 + 16\,\varepsilon^2 n(n-1)}).$

The Eigen values are roots of the polynomials

$P(X) = X^2 - \rho X - \mathcal{E}\,n\,(\rho + 2\,\mathcal{E}\,n) + \frac{n}{s^2}$ …………..(20)

We calculate $\Delta = B^2 - 4AC$ in standard quadratic equations $AX^2 + BX + C=0$

Here in (20) A = 1, B = - ρ ,C = $-2\,\mathcal{E}\,n\,(\rho + 2\,\mathcal{E}\,n) + \frac{n}{s^2}$

Therefore $\Delta = (-\rho)^2 - 4(1)[-2\mathcal{E}\,n\,(\rho + 2\,\mathcal{E}\,n) + \frac{n}{s^2}$].

$= \rho^2 + 8\mathcal{E}n\,[(\rho + 2\,\mathcal{E}\,n) - \frac{1}{2\sigma^2}$]

Using the relative $\mathcal{E} = \frac{1}{\sigma^2} - \frac{\rho}{2}$ we get

⟹ $\sigma^2 = \frac{2}{2\,\mathcal{E} + \rho}$

$\Delta = \rho^2 + 8n\left[\frac{1}{\sigma^2} - \frac{\rho}{2}\right]\left[\rho + 2n\left\{\frac{1}{\sigma^2} - \frac{\rho}{2}\right\} - \frac{1}{2\sigma^2}\right]$

$= \rho^2 + \left(\frac{8n}{\sigma^2} - 4np\right)\left[\rho + \frac{2n}{\sigma^2} - np - \frac{1}{2\sigma^2}\right]$

$\Delta = \rho^2 + 16\,\varepsilon^2 n(n-1)$ ………………(21)

Since n≥2, when n = 2 ⟹ $\Delta = \rho^2 + 32\,\varepsilon^2$ and

n = 3 ⟹ $\Delta = \rho^2 + 96\varepsilon^2$

we have $\Delta > \rho^2$.

ie, The two roots are real , one is positive and the other one is negative.

4.3. Proposition IV:

Let us suppose the two perturbations $\dot{\phi}_n$ and $\dot{\psi}_n$ are in the Hilbert space

H= $\mathcal{L}^2$[m*(x)dx] and n ≥ 2 then the functions $\overline{\phi}$ and $\underline{\psi}$ are in the Hilbert

$\begin{bmatrix} \dot{\phi}_n \\ \dot{\psi}_n \end{bmatrix} = A_n \begin{bmatrix} \phi_n \\ \psi_n \end{bmatrix}$

With $\phi_n(T) = \overline{\phi_n}$

and $\psi_n(o) = \underline{\psi_n}$

Solution procedure:

From the proof of proposition III we can write

$\begin{bmatrix} \phi_n(t) \\ \psi_n(t) \end{bmatrix} = A_{n,T,}\, e^{\lambda_{n,T}^1}\begin{bmatrix} 1 \\ v_n^1 \end{bmatrix} + B_{n,T,}\, e^{\lambda_{n,T}^2}\begin{bmatrix} 1 \\ v_n^2 \end{bmatrix}$

Where v_n^1 and v_n^2 are Eigen vectors of the matrix A_n.

ie , $v_n^1 = \rho + 2\,\mathcal{E}\,n - \lambda_n^1$

$v_n^2 = \rho + 2\,\mathcal{E}\,n - \lambda_n^2$

Next we using the conditions on $\phi_n(T)$, $\psi_n(o)$ to compose the constant $A_{n,T}$ & $B_{n,T}$

$$\left.\begin{aligned} \phi_n(T) &= \overline{\phi_n} = A_{n,T}\,\gamma\, e^{\lambda_{n,T}^1} + B_{n,T}\,\gamma\, e^{\lambda_{n,T}^2} \\ \psi_n(o) &= \underline{\psi_n} = A_{n,T}\,\gamma\, v_n^1 + B_{n,T}\,\gamma\, v_n^2 \end{aligned}\right\} \quad ……….(22)$$

Hence we get

$A_{n,T} = \frac{v_n^2\overline{\phi_n} - e^{\lambda_{n,T}^2}\underline{\psi_n}}{v_n^2 e^{\lambda_{n,T}^1} - v_n^1 e^{\lambda_{n,T}^2}}$ & $B_{n,T} = \frac{v_n^1\overline{\phi_n} - e^{\lambda_{n,T}^1}\underline{\psi_n}}{v_n^1 e^{\lambda_{n,T}^2} - v_n^2 e^{\lambda_{n,T}^1}}$ …. (23)

Using this relation

$v_n^1 \sim 4\,\mathcal{E}\,n$ and

$v_n^2 \sim \frac{\rho}{2} + \mathcal{E}$

Then we get with T fixed as n→∞

$A_{n,T} \sim$ n→∞ $\frac{\underline{\psi_n}}{4\,\mathcal{E}\,n}$ & $B_{n,T} \sim$ n→∞ $\overline{\phi_n}\, e^{-\lambda_{n,T}^2}$

Hence $\phi_n(t) = \mathcal{O}_n\left[\frac{\underline{\psi_n}}{4\,\mathcal{E}\,n} e^{\lambda_{n,t}^1}\right] + \mathcal{O}_n\left[\overline{\phi_n} e^{-\lambda_n^2 (T-t)}\right]$ ... (24)

$\psi_n(t) = \mathcal{O}_n\left[\underline{\psi_n} e^{\lambda_{n,t}^1}\right] + \mathcal{O}_n\left[\overline{\phi_n} e^{-\lambda_n^2 (T-t)}\right]$ ……(25)

By combining the above two calculations prove the results.

V. CONCLUSION

This paper presents a detailed study of Mean Field Game Theory and associated concepts. The classifications of some Frame Works in Economics and in Eigen values approach have been discussed. Also, a dynamical Mean Field Game model in continuous time with a continuous state space. Further we have deal with two relevant calculations of Eigen vectors approach to solve PDE systems equation. Finally, this paper gives an insight to the use of Mean Field Game applications derived.

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