On Triangular Sum Graphs of some Graphs

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***Abstract*: A triangular sum labelling of a graph G is a one – to – one function f: V → N (where N is the set of all non – negative integers) that induces a bijection on f^+: E (G) →{T_1, T_2, ...,T_q} of the edges of G defined by $f^+(uv) = f(u) + f(v)$, ∀uv ϵE(G).In this paper we proved the following graphs star graphs $S^+(n,m)$, Generalized Butane graph, n – Centipede union P_n, Fork graph are *triangular sum graphs*.**

***Keywords*: Star graph, Path graph Generalized Butane graph.**

Introduction:

In 2008 S.Hedge and P.Shankaran [1] call a labelling of graph with q edges a triangular sum labelling if the vertices can be assigned distinct non – negative integers in such a way that, when an edge whose vertices are labelled i and j is labelled with the value i + j , the edge labels are {k(k+1)/2} where k = 1,2,...q. In this paper we can see some star linked graphs and named graphs are triangular sum graph.

Definition: 1.1
A triangular number is a number obtained by adding all positive integers less than or equal to a given positive integer n. If n^{th} triangular number is denoted by T_n then $T_n = \frac{1}{2}n(n+1)$. It is easy to observe that there does not exist consecutive integers which are **triangular numbers**.

Definition:1.2
A triangular sum labelling of a graph G is a one – to – one function f: V → N (where N is the set of all non – negative integers) that induces a bijection on f^+: E (G) →{T_1, T_2, ...,T_q} of the edges of G defined by $f^+(uv) = f(u) + f(v)$, ∀uv ϵE(G).

Some Known Results:

S.Hedge and Shankaran proved the following results:

1. Path P_n, Star $K_{1,n}$, are triangular sum graphs.
2. Any tree obtained from the star $K_{1,n}$ by replacing each edge by path is a triangular sum graph.
3. The lobster T obtained by joining the centers of k copies of a star to a new vertex w is a triangular sum graph.
4. The Complete n – array graph T_m of level m is a triangular sum graph.
5. The Complete graph K_n is triangular sum graph iff n ≤ .
6. If G is an Eulerian (p,q) – graph admitting a triangular sum labelling then q ≠1(mod 12).
7. The Dutch windmill DW(n) (n copies of K_3 sharing a common vertex) is not a triangular sum graph.
8. The Complete graph K_4 can be embedded as an induced subgraph of a triangular sum graph.

Theorem:1.1

Let S_n be a star with n + 1 vertices. Let G be the disjoint union of m copies of S_n. Then G is triangular sum graph.

Proof

Let {a_0, a_1, a_2,...,a_n} be the vertices of the star S_n. Consider m isomorphic copies of S_n. Let G is the disjoint union of m copies of S_n.

Let V(G) = {a_{ij}/ 1≤ i ≤ n + 1 ; 1 ≤ j ≤ m}. Note that the graph G has mn edges and m(n + 1) vertices.

Define f : V(G) → {0, 1, 2,...,T_{mn}} as follows.

Now label the vertex $f(a_{11})$ as 0 and $f(a_{1j})$,j = 2,3,...,n +1 as T_1,T_2,...,T_n respectively so as the edges $f(a_{11}a_{1j})$, j = 2,3,...,n +1 must obtain the values as T_1,T_2,...,T_n.

In the second copy, let a_{22}, a_{23}, ...,$a_{2(n+1)}$ be the vertices adjacent to a_{21}. Label these vertices as $f(a_{21})$ =1 and others as T_{n+i} -1, 1 ≤ i ≤ n. So as the edges $f(a_{21}a_{2j})$, 2 ≤ j ≤ n + 1 obtain the values by $f(a_{2j}) + f(a_{2i}) = T_{n+i}$, 1 ≤ i ≤ n.

Next let a_{32}, a_{33},...,$a_{3(n+1)}$) be the vertices adjacent to a_{31} in the third copy . Label the vertices as $T_{2n+i} - 2$, $1 \le i \le n$ and $f(a_{31}) = 2$ from this we obtain the edge labels as $f(a_j) + f(a_{31}) = T_{2n+i}$, $1 \le i \le n$.

Proceeding likethis we get in the m^{th} copy of the graph G has the vertex set $a_{m1}, a_{m2}, \ldots, a_{m(n+1)}$. Labels the vertices as $f(a_{m1}) = m - 1$ and corresponding other vertices $T_{(m-1)n+i} - (m - 1)$, $1 \le i \le n$ and the corresponding edge labels are $T_{mn-(n-1)}, \ldots, T_{mn-1}, T_{mn}$.

Clearly all the vertex labelings are distinct and edge values are in the form $\{T_1, T_2, \ldots, T_{mn}\}$. This completes the proof. Hence G is triangular sum graph.

Example:1

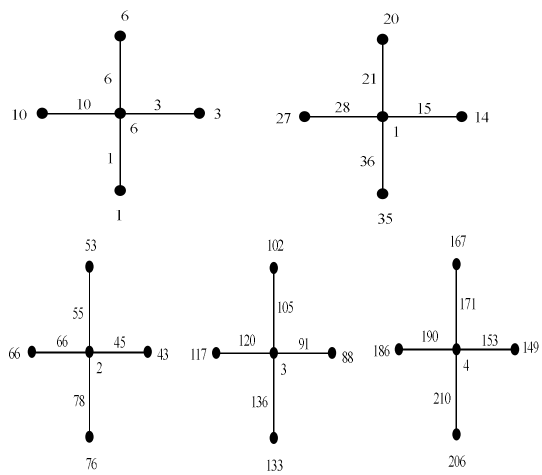

Fig:1 **5 copies of S_4 is triangular sum graph.**

Definition:1.3

Let S_n be a star with $(n + 1)$ vertices. Consider m copies of S_n. Identify any one vertex of the i^{th} copy other than the central vertex with any one vertex other than the centre vertex of $(i + 1)^{th}$ copy, the graph so obtained is denoted as $S^+(n,m)$.

Theorem:1.2

$S^+(n,m)$ is triangular sum graph for all $n \ge 3$ and m.

Proof:

Let $\{a_{ij} / 1 \le i \le n\ 1, 1 \le j \le m\}$ be the vertex set of m copies of S_n . Then one vertex of the i^{th} copy other than the central vertex with any one vertex other than the centre of $(i + 1)^{th}$ copy.

Here we join the vertex of m copies to $a_{(i+1)}{}^{th}$ vertices. The graph has $(mn + 1)$ vertices and mn edges.

Define $f : V(G) \rightarrow \{0,1,2,\ldots,T_{mn}\}$ as follows:

$$f(a_{11}) = 0$$

$$f(a_{1j}) = T_i,\ 1 \le i \le n$$

$$f(a_{21}) = T_{n+1} - T_1$$

$$f(a_{2j}) = T_{n+i} - f(a_{21}),\ 2 \le i \le n$$

$$f(a_{31}) = T_{2n+1} - f(a_{2n})$$

$$f(a_{3j}) = T_{2n+i} - f(a_{3j}),\ 2 \le i \le n$$

. . .

. .

$$f(a_{m1}) = T_{(m-1)n+1} - f(a_{(m-1)n})$$

$$f(a_{mj}) = T_{(m-1)n+i} - f(a_{(m-1)n}),\ 2 \le i \le n$$

Clearly the vertex labels are distinct.

Now, from the definition , the edge values are

$$f(a_{1j}) + f(a_{11}) = T_i,\ 1 \le i \le n,\ 2 \le j \le n + 1$$

$$f(a_{21}) + f(a_{12}) = T_{n+1}$$

$$f(a_{2j}) + f(a_{21}) = T_{n+i},\ 2 \le i \le n - 1,\ 2 \le j \le n + 1$$

$$f(a_{31}) + f(a_{21}) = T_{2n+1}$$

$$f(a_{3j}) + f(a_{31}) = T_{2n+i},\ 2 \le i \le n - 1,\ 2 \le j \le n + 1$$

. . .

. . .

$$f(a_{m1}) + f(a_{(m-1)1}) = T_{(m-1)n+1}$$

$$f(a_{mj}) + f(a_{m1}) = T_{(m-1)n+i},\ 2 \le i \le n - 1,\ 2 \le j \le n + 1$$

Also $f(a_{mn}) + f(a_{m1}) = T_{mn}$.

Hence the edge values are in the form $\{T_1, T_2, \ldots, T_{mn}\}$. Thus $S^+(n, m)$ is a triangular sum graph.

Example:2

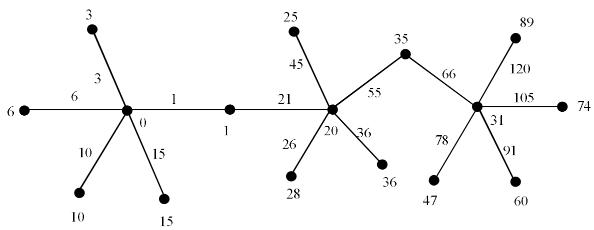

Fig: 2 **S^{+}(5,3) is triangular sum graph.**

Theorem: 1.3

Fork graph is triangular sum graph.

Proof

Let G be a graph with 5 vertices and 4 edges. Let the vertex set be $V(G) = \{u_i / 1 \leq i \leq 5\}$.

Let the edge set be $E(G) = \{u_iu_{i+1} / 1 \leq i \leq 2\} \cup \{u_1u_4\} \cup \{u_4u_5\}$.

Define $f : V(G) \rightarrow \{0,1, 2,...,T_4\}$ such that

$f(u_1) = 0$

$f(u_2) = 3$

$f(u_3) = 6$

$f(u_4) = 1$

$f(u_5) = 9$

Clearly the vertex labels are distinct. Hence f is injective and the edge labels are of the form $\{T_1,T_2,...,T_4\}$ is given by

$f^*(e_1) = 3 = T_2$

$f^*(e_2) = 6 = T_3$

$f^*(e_3) = 1 = T_1$

$f^*(e_4) = 10 = T_4$

clearly all the vertex labels are distinct and the edge labels are of the form

$\{T_1,T_2,...,T_4\}$. Hence fork graph is triangular sum graph.

Example: 3

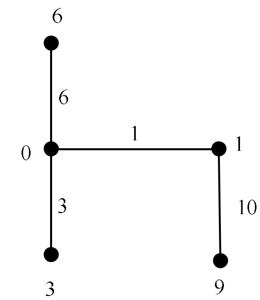

Fig:3 **Fork graph is triangular sum graph.**

Definition:1.4

Generalized Butane graph

Generalized Butane graph is defined as follows. Let G be a graph with $V(G) = \{u_i / 1 \leq i \leq n\} \cup \{w_i / 0 \leq i \leq n+1\}$ and $E(G) = \{u_iw_i / 0 \leq i \leq n\}$. Then the graph G has 3n +1 edges.

Theorem:1.4

Generalized Butane graph is triangular sum graph.

Proof

Let G be the graph with $V(G) = \{u_i / 1 \leq i \leq n\} \cup \{w_i / 0 \leq i \leq n + 1\}$ and $E(G) = \{u_iw_j / 1 \leq i \leq n\} \cup \{w_iv_i / 1 \leq i \leq n\} \cup \{w_iw_{i+1} / 0 \leq i \leq n\}$. Then the graph G has 3n + 2 vertices and 3n + 1 edges.

Define $f: V(G) \rightarrow \{0, 1, 2,...,3n +1\}$ as follows.

Now label the vertex $f(w_1)$ as 0 and

$f(w_2) = T_1$

$f(w_3) = |T_1 - T_2|$ and

$f(w_i) = |T_{i-2} - T_{i-1}|$, $4 \leq i \leq n + 1$.

So as the edges $w_1w_2,w_2w_3,...,w_nw_{n+1}$ must obtain the values as $T_1,T_2,...,T_n$.

Next let $u_1,u_2,...,u_n$ be the vertices adjacent to $w_1,w_2,...,w_n$ in left. Label the vertices $f(u_i)$ as $|T_{n+i} - f(w_i)|$, $1 \leq i \leq n$ and so as the edges $u_1w_1,u_2w_2,...,u_nw_n$ must obtain the values as T_{n+i}, $1 \leq i \leq n$.

Also let $v_1, v_2, \ldots, v_n$ be the vertices adjacent to $w_1, w_2, \ldots, w_n$ in right. Label the vertices $f(v_1)$ as T_{3n} and $v_2, \ldots, v_n$ as$|T_{3n-i} - f(w_{i+1})|$, $1 \le i \le n - 1$. And the corresponding edges $v_1w_1, v_2w_2, \ldots, v_nw_n$ must obtain the values as T_{3n-i} for $0 \le i \le n - 1$.

Also the vertex $f(w_0)$ has $3n + 1$ and the corresponding edge $f(w_0w_1) = 3n + 1$.

Clearly, all the vertex labelings are distict and edge values are in the form $\{T_1, T_2, \ldots, T_{3n+1}\}$. This completes the proof.

Hence Generalized Butane graph is triangular sum graph.

Example: 4

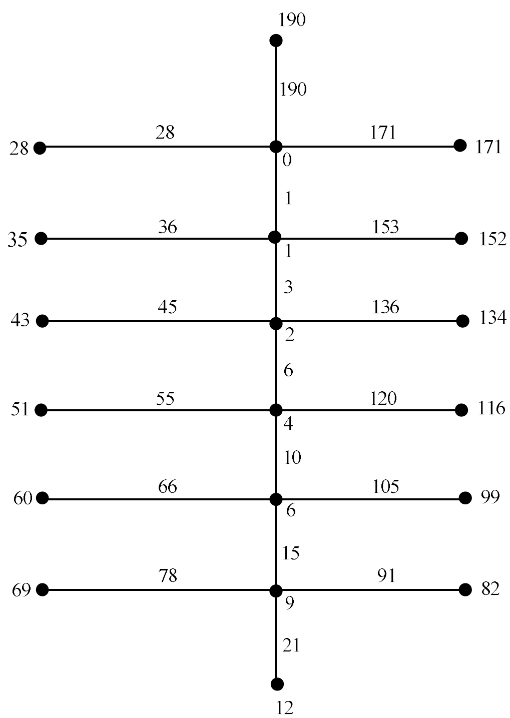

Fig: 4 **Generalized Butane graph of n = 6 is triangular sum graph.**

Theorem:1.5

n – Centipede union P_n is triangular sum graph.

Proof

The n – Centipede is the tree on 2n nodes obtained by joining the bottoms of n copies of the path graph P_2 laid in a row with edges. It has 2n vertices and 2n – 1 edges.

The path graph P_n is of n vertices and n – 1 edges.

Let G_1 be the n – centipede u_i and v_i, $1 \le i \le n$ and G_2 be the path P_n of $w_1, w_2, \ldots, w_n$. Then $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$. Now the graph G has 3n vertices and 3n – 2 edges.

Define $f: V(G) \rightarrow \{0,1,2,3,\ldots,T_{3n-2}\}$ as follows. Now label the vertices as follows

$$f(u_1) = 1$$

$$f(u_i) = |T_i - f(u_{i-1})|, 2 \le i \le n$$

From this we get the edges $u_1u_2, u_2u_3, \ldots, u_{n-1}u_n$ must obtain the values $T_2, T_3, \ldots, T_n$.

Also the vertex labels of v_i are

$$f(v_1) = 0$$

$$f(v_i) = |f(u_i) - T_{2i-i}|, 2 \le i \le n$$

so as from above results u_1v_1 must obtain the value T_1 and the remaining edges $u_2v_2, u_3v_3, \ldots, u_nv_n$ must obtain $T_{2n-2}, T_{2n-3}, \ldots, T_{n+1}$.

Also the vertex label of w_i, $1 \le i \le n$ by

$$f(w_1) = 3$$

$f(w_i) = |f(w_{i-1}) - T_{2n+j}|$, $2 \le i \le n$, $0 \le j \le n - 2$ and so the edges w_iw_{i+1} , $1 \le i \le n - 1$ must obtain the values $T_{2n}, T_{2n+1}, \ldots, T_{3n-2}$.

Clearly all the vertex labels are distinct and the edge values are in the form $\{T_1, T_2, \ldots, T_{3n-2}\}$. This completes the proof. Hence G is triangular sum graph.

Example: 5

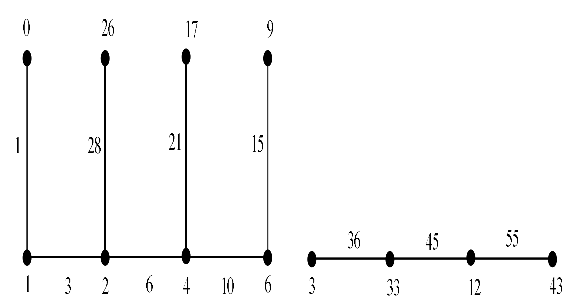

Fig: 5 **4 – Centipede union P_4 is triangular sum graph.**

Definition:1.5

A Y- tree Y_{n+1} is a graph obtained from the path P_n by appending an edge to a vertex of the path P_m adjacent to an end point.

Theorem:1.6

For $r \geq 3$, the Y – tree Y_{n+1} is a triangular sum graph.

Proof

Let $V(G) = \{u_1,u_2,...,u_n\} \cup \{v\}$ and $E(G) = \{u_iu_{i+1}/\ 1 \leq i \leq n - 1\} \cup \{u_{n-1}v\}$.

Note that the graph Y – tree has n + 1 vertices and n edges.

Define $f : V(G) \rightarrow \{0,1,2,...,T_n\}$ as follows

$f(u_1) = 1$

$f(u_2) = 0$

$f(u_3) = 6$

$f(u_i) = |f(u_{i-1}) - T_i|,\ 4 \leq i \leq n$

$f(v) = 3$

clearly all the vertex labels are distinct and the edge values are

$f(u_1u_2) = 1$

$f(u_2u_3) = 6$

$f(u_iu_{i+1}) = T_{i+1},\ 3 \leq i \leq n$

$f(u_2v) = 3$

Therefore the edge values are in the form $\{T_1,T_2,...,T_n\}$. Hence the proof.

Example: 6

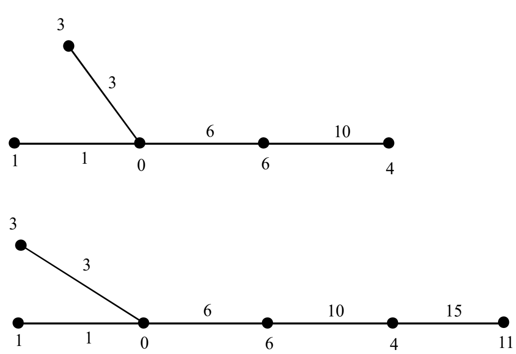

Fig: 6 **Y_{4+1}, Y_{5+1} are triangular sum graph.**

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